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On the essential spectrum of the Laplacian on complete manifolds

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1. Introduction.

The Laplace-Beltrami operator Δ on a noncompact complete Riemannian manifold M is essentially self-adjoint on $C_0^\infty(M)$, and the spectrum of its self-adjoint extension to $L^2(M)$ has been studied by several authors from various points of view. For instance, Donnelly [**6**] proved that the essential spectrum $\sigma_{ess}(-\Delta)$ of $-\Delta$ is equal to $[(n-1)^2k^2/4, \infty)$ if M is an n-dimensional Hadamard manifold whose sectional curvatures approach a constant $-k^2$ at infinity. On the other hand, Escobar and Freire [**5**] consider the case M has nonnegative sectional curvatures and showed that $\sigma_{ess}(-\Delta)=[0, \infty)$, if M possesses a soul S such that the normal exponential map $\exp_S^\perp : NS \to M$ induces a diffeomorphism, and further if either $\dim M=2$, or $\dim M \geqq 3$ and $\int_1^\infty (1/v(t))\left[\int_{S_t} \mathrm{Ric}(\nabla r)\right]dt<\infty$, where NS is the normal bundle to S, $r(x)=\mathrm{dist}(x, S)$, $S_t=\{x\in M \mid \mathrm{dist}(x, S)=t\}$, and $v(t)$ denotes the volume of S_t. Recently Li [**10**] proved that if M has nonnegative Ricci curvatures and possesses a pole (i.e., a point $x\in M$ where the exponential map $\exp_x : T_xM \to M$ induces a diffeomorphism), then $\sigma_{ess}(-\Delta)$ is equal to $[0, \infty)$.

In this paper, we shall show the following:

THEOREM 1.1. *Let M be a noncompact complete Riemannian manifold of dimension n. Suppose there exists an open subset U of M with compact smooth boundary ∂U such that the outward-pointing normal exponential map $\exp_{\partial U}^\perp$: $N^+(\partial U) \to M-\bar{U}$ induces a diffeomorphism. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function satisfying*

$$\mathrm{Ric}_M(\gamma_x'(t), \gamma_x'(t)) \geqq -(n-1)\varphi(t),$$

for all $t\geqq 0$ and $x\in\partial U$, where $\gamma_x(t)=\exp_x(t\cdot\vec{n}(x))$ and $\vec{n}$ stands for the outward unit normal vector field on ∂U. Then the spectrum $\sigma(-\Delta)$ of $-\Delta$ is equal to $[0, \infty)$, provided that $\varphi(t)$ converges to zero as $t\to\infty$.

This is a generalization of the results by Escobar and Freire, and also Li mentioned above, and it will be deduced from a comparison argument and the following