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Geometry of weakly symmetric spaces

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1. Introduction.

Weakly symmetric spaces have been introduced by A. Selberg [**21**] in 1956. His motivation was to generalize the Poisson summation formula to what is now known as the Selberg trace formula. These homogeneous spaces have the property that the differential operators which are invariant under the action of the full isometry group form a commutative algebra, that is, these spaces are commutative. Whereas much work has been done on the harmonic analysis of the commutative and weakly symmetric spaces, in particular on $SL(2, \boldsymbol{R})$, the geometry of the weakly symmetric spaces has to our knowledge not been studied thoroughly. Only a few properties are given in [**3**]. The reason might be that Selberg's definition of weakly symmetric spaces appears to be rather abstract and only a few examples are known. The intention of this note is to point out that there is a nice geometrical characterizing property hidden in Selberg's definition leading to the construction of a whole list of new examples, and to stimulate further research on the Riemannian geometry of this class of spaces.

A Riemannian manifold M is said to be a *weakly symmetric space* if there exists a subgroup G of the isometry group $I(M)$ of M acting transitively on M and an isometry f of M with $f^2 \in G$ and $fGf^{-1}=G$, such that for any two points $p, q \in M$ there exists an isometry $g \in G$ with $g(p)=f(q)$ and $g(q)=f(p)$. By taking $G=I(M)$ and $f=id_M$ it can be seen easily that any Riemannian globally symmetric space is weakly symmetric. The crucial observation for our studies is the following

GEOMETRICAL CHARACTERIZATION OF WEAKLY SYMMETRIC SPACES. *A Riemannian manifold M is weakly symmetric if and only if for any two points p, q in M there exists an isometry of M mapping p to q and q to p.*

The proof is elementary. First, suppose M is weakly symmetric and p, q are any two points in M. Let $g \in G$ be an isometry of M with $g(p)=f(q)$ and $g(q)=f(p)$. Then $f^{-1}g$ is an isometry of M mapping p to q and q to p. To see the converse, put $G=I(M)$ and $f=id_M$.