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Factors generated by direct sums of II_1 factors

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Introduction.

In 1983 Jones introduced in [**3**] the concept of an index for a pair of type II_1 factors, called Jones index nowadays, and he showed the importance of such indices. With this as a momentum, the interests of research in the theory of operator algebras have been gradually extended from a single factor to a pair of factors. Thereafter Pimsner-Popa [**8**] gave an important relation between the index and the relative entropy for a pair of finite von Neumann algebras and showed that if $N\subset M$ is a pair of II_1 factors with finite index, then there exists a certain orthonormal basis of M over N. In the case of type III factors, Kosaki [**4**] defined an index depending on a conditional expectation and, on the other hand, Longo [**5**] gave another definition by using the canonical endomorphism. And in the case of C^*-algebras, Watatani [**12**] defined an index by using a quasi-basis.

However it is not easy to calculate explicitly the index even for a pair of II_1 factors only from the definition. For this reason, useful index formulas are expected. So far, Pimsner-Popa [**8**], Wenzl [**13**] and Ocneanu [**7**] gave index formulas respectively. Wenzl's formula is applicable only for pairs of approximately finite dimensional (=AFD) II_1 factors. In this paper we give a new index formula, that is the extension of Wenzl's one, and its application, for a pair of II_1 factors which are not necessarily AFD.

We treat a pair of II_1 factors arising from two increasing sequences of finite direct sums of II_1 factors. Let us explain more exactly, denote the sequences by $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$, and assume that the diagram

$$\text{(A)}\qquad \begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square for any n. Set $M=(\bigcup_n M_n)''$ and $N=(\bigcup_n N_n)''$. If the inclusion relations $N_n\subset N_{n+1}$, $M_n\subset M_{n+1}$ and $N_n\subset M_n$ are periodic, then M and N are found to be II_1 factors. For such a pair $N\subset M$ we give an index formula.