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On a normal integral bases problem over cyclotomic Z_p-extensions

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§ 1. Introduction.

Let p be a prime number and K be a number field containing a primitive p-th root of unity. Let $\mathcal{H}(K)$ be the subgroup of $K^{\times}/K^{\times p}$ consisting of elements $[\alpha]$ ($\in K^{\times}/K^{\times p}$) for which the extension $K(\alpha^{1/p})$ is unramified over K, and $\mathcal{N}(K)$ be the subset of $\mathcal{H}(K)$ consisting of elements $[\alpha]$ ($\in \mathcal{H}(K)$) for which the unramified cyclic extension $K(\alpha^{1/p})/K$ has a relative normal integral bases. Here, we say that a Galois extension L/E of a number field E has a relative normal integral bases (an RNIB, for short) when the integer ring O_L of L is free over the group ring $O_E[\mathrm{Gal}(L/E)]$. In [3], Childs gave a criterion for a cyclic extension L/K of degree p to be unramified and have an RNIB (see Lemma 5 in § 4), from which it follows that $\mathcal{N}(K)$ is a subgroup of $\mathcal{H}(K)$. He raised the question "what is the quotient group $\mathcal{H}(K)/\mathcal{N}(K)$?". We have been investigating this problem for certain abelian fields ([**14**], [**15**]) in connection with power series associated to certain p-adic L-functions. A similar study is also given in Taylor [**24**] when K is the p-th cyclotomic field $\boldsymbol{Q}(\mu_p)$. In this paper, we shall continue these investigations.

Let p be an odd prime number and k be an imaginary abelian field satisfying the following conditions:

(C1) k contains a primitive p-th root of unity.
(C2) $p \nmid [k:\boldsymbol{Q}]$.
(C3) There is only one prime ideal of k over p.

Let k_{∞}/k be the cyclotomic $\boldsymbol{Z}_p$-extension and k_n ($n \geqq 0$) be its n-th layer. We write, for brevity, $\mathcal{H}_n = \mathcal{H}(k_n)$ and $\mathcal{N}_n = \mathcal{N}(k_n)$. The Galois groups $\Delta = \mathrm{Gal}(k/\boldsymbol{Q})$ and $\Gamma = \mathrm{Gal}(k_{\infty}/k)$ act on these groups in a natural way. In particular, we may decompose these groups by the action of complex conjugation ρ ($\in \Delta$); $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$, $\mathcal{N}_n = \mathcal{N}_n^+ \oplus \mathcal{N}_n^-$. As far as normal integral bases problem is concerned, we have nothing to consider on the "odd" part, because we already know that $\mathcal{N}_n^- = \{1\}$ (Brinkhuis [**1**]). As for the "even" part, we have described,

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