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Identifying tunnel number one knots

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Let K be a knot in S^3. The tunnel number $t(K)$ of K is the minimal number of mutually disjoint arcs $\{\tau_i\}$ "properly embedded" in the pair (S^3, K) such that the complement of an open regular neighbourhood of $K\cup(\cup\tau_i)$ is a handlebody. In the above, if the arc system consists of only one arc, it is called an *unknotting tunnel* for K. K is said to have a (g, b)*-decomposition* if there is a genus g Heegaard splitting $\{W_1, W_2\}$ of S^3 such that K intersects W_i $(i=1, 2)$ in a b-string trivial arc system (cf. [**D**, **MS**]). If a knot K has a (g, b)-decomposition, then $t(K)\leqq g+b-1$. In particular; if K admits a (1, 1)-decomposition then it has tunnel number one; however, it is shown by [**MR**, **MSY**, **Yo1**] that the converse does not hold.

Kohno [**Kh**] gave an estimate of tunnel numbers of knots in terms of the quantum invariants (cf. [**Wk**, **G**]), and the third author [**Yo1**] gave a condition for a knot to admit a (g, b)-decomposition in terms of the quantum $SU(2)$ invariants. Kouzi Kodama [**Kd**] applied Kohno's estimate to prime knots up to 10 crossings by using his computer program "Knot", and determined the tunnel numbers of several such knots.

In this paper, we give another method to determine whether a given knot K has tunnel number one and whether it admits a (1, 1)-decomposition, by using the idea due to Birman-Hilden [**BH**] and Viro [**V**] (cf. [**BGM**], [**BM**], [**BoZe**]). The method enables us to determine the tunnel numbers of prime knots up to 10 crossings (Theorem 2.5), and is potentially useful to the problem of detecting tunnel number one knots which do not admit (1, 1)-decompositions. The idea is to look at the canonical 2-fold symmetry arising from an unknotting tunnel and to reduce the problem to that concerning symmetries of knots and that concerning spatial θ-curves (Theorem 1.2). Study of symmetries of knots has long history, and we now have enough information concerning symmetries of various kinds of knots, including the Montesinos knots and the prime knots up to 10 crossings (see [**AHW**, **BoZm**, **HW**, **KS**]). On the other hand, there is a naive but convenient method for the study of the problem concerning spatial θ-curves (Corollary 1.3). By using this method, we obtain a certain condition for a