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Jacobi sums and the Hilbert symbol for a power of two(*)

By Hiroo MIKI

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Many number theorists have taken up the problem of determining the exact conductor $C_m^{(a)}$ of the Jacobi sum Hecke character $\mathfrak{a}\mapsto J_m^{(a)}(\mathfrak{a})$ since Weil [**18**] raised its interesting problem in 1952. Recently, Coleman-McCallum [**2**] determined the exact conductor $C_m^{(a)}$ when m is a power of any *odd* prime number l, using the arithmetic geometry of Fermat curves, and Miki [**12**], [**13**], [**14**] gave a purely number theoretic proof to their results. But the case $l=2$ is still an unsolved more difficult open problem, and it seems that Coleman-McCallum's method [**2**] is not applicable to the case $l=2$, though Coleman [**3**], § 6 (with G. Anderson) gave a partial result by using Ihara-Anderson's theory.

The purpose of the present paper is to give the complete determination of the conductor $f_n(g, h, s)$ of the character $\alpha\mapsto(\alpha, 2^g(1+4)^h(-1)^s)_n$ with $g\in \boldsymbol{Z}$, $h\in \boldsymbol{Z}_2$, and $s\in \boldsymbol{Z}/2\boldsymbol{Z}$ for $n\geqq 2$ (see Theorem 5 in § 1), and the conductor $C_{2^n}^{(a)}$ of the Jacobi sum Hecke character $\mathfrak{a}\mapsto J_{2^n}^{(a)}(\mathfrak{a})$ for the power 2^n (see Corollary to Theorem 9 in § 2), by the methods of [**13**], [**14**]. Here, $\boldsymbol{Z}$ and $\boldsymbol{Z}_2$ are the rings of rational and 2-adic integers respectively, and $(\ ,\ )_n$ denotes the Hilbert norm residue symbol in $\boldsymbol{Q}_2(\zeta_{2^n})$ for the power 2^n, where $\boldsymbol{Q}_2$ is the field of 2-adic numbers and ζ_{2^i} is a fixed primitive 2^i-th root of unity satisfying $\zeta_{2^{i+1}}^2=\zeta_{2^i}$ for all $i\geqq 1$ (for the exact definition, see [**14**], § 1).

Since $\delta^{(n)}(\alpha)$ is well-defined mod 2^{n-1} (not mod 2^n) when $l=2$ (see Lemma 6 in § 2), we can determine $i_{2^n}^{(a)}(\alpha)$ mod 2^{n-1} in the same way as [**13**] (see Theorem 8 in § 2). In Theorem 9 (see also its Remark) in § 2, we will determine $i_{2^n}^{(a)}(\alpha)$ mod 2^n for $\alpha\in \boldsymbol{Q}(\zeta_{2^n})$, $\alpha\equiv 1 \pmod{\pi_n^3}$, by using Theorem 8 and certain congruences for Jacobi sums (see Theorems 12, 13, and 14 in § 3). Note that Theorem 9 (and its Remark) contains Coleman [**3**], Theorem (6.4) as a special case. Theorem 9, combined with Theorem 5, gives the complete determination of the conductor $C_{2^n}^{(a)}$ (see Corollary to Theorem 9).

(*) This paper contains the details of part of my talk at the Number Theory Seminar (Goldfeld), Columbia Univ., March 21, 1988 (see [**12**]).