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On the quantization of a coherent family of representations at roots of unity

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1.

Given a coherent family of virtual representations of a complex semisimple Lie algebra we associate a coherent family of virtual representations of the corresponding quantum group at roots of unity. The latter family depends on the given family in a precise fashion described below.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and let U be its universal enveloping algebra. Lusztig considered a certain $\boldsymbol{C}[v, v^{-1}]$ algebra $U_{\mathcal{A}}$, $\{\mathcal{A}=\boldsymbol{C}[v, v^{-1}]\}$ which is an $\mathcal{A}$-form of the 'quantum group' $U_{\mathcal{A}'}$, $\{\mathcal{A}'=\boldsymbol{C}(v)$, the field of fractions of $\mathcal{A}\}$; the latter are some Hopf-algebra deformations of U, defined by Drinfeld and Jimbo generalizing the case of $\mathfrak{sl}_2$.

Let $\lambda\in\boldsymbol{C}^*$ and suppose that λ is a primitive l-th root of unity where l, $(\geqq 3)$, is an odd positive integer (not divisible by 3 if G_2 is a factor of $\mathfrak{g}$).

Let $\varphi_\lambda : \mathcal{A}\rightarrow\boldsymbol{C}$ be the $\boldsymbol{C}$-algebra homomorphism obtained by sending v to λ. The algebras $U_\lambda :=U_{\mathcal{A}}\otimes_{\mathcal{A}}\boldsymbol{C}$, (scalar multiplication by $u\in\mathcal{A}$ in the first factor corresponds to scalar multiplication by $\varphi_\lambda(u)$ in the second factor) are called 'quantum groups at roots of unity' ; these are different from those considered by Kac, Processi and De-Concini. The algebras U_λ are also Hopf algebras.

In [**L1**, Prop. 7.5(a) and **L2**, 8.16] Lusztig defines a 'Frobenius' morphism $\psi: U_\lambda\rightarrow U$; ψ is a surjection and respects the Hopf-algebra structure.

2. Coherent family of virtual representations of U, U_λ.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Δ^+ a system of positive roots. Let $S=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the set of simple roots in Δ^+. Let $\Lambda\subseteqq\mathfrak{h}^*$ $(=\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C}))$ be the integral lattice defined by

$$\nu\in\Lambda \Longleftrightarrow 2(\nu, \alpha)/(\alpha, \alpha)\in\boldsymbol{Z}, \qquad \forall\alpha\in\Delta$$

where the pairing is induced by the Killing form in the usual way.