J. Math. Soc. Japan
Vol. 48, No. 1, 1996

On the filtration of topological and pro-l mapping class groups of punctured Riemann surfaces

By Mamoru ASADA

(Received Feb. 14, 1994)

Introduction.

Let $R_{g,n}$ be a Riemann surface of genus $g \geqq 0$ with $n \geqq 0$ punctures, and $\Gamma_{g,n}$ be the pure mapping class group of $R_{g,n}$. As is well known, $\Gamma_{g,n}$ is isomorphic to a certain subgroup of the outer automorphism group of the topological fundamental group $\pi_{g,n}$ of $R_{g,n}$. The group $\pi_{g,n}$ has a natural filtration $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ called the weight filtration, which is introduced by T. Oda. This filtration naturally induces a filtration $\{\Gamma_{g,n}[m]\}_{m=0}^{\infty}$ of $\Gamma_{g,n}$. The aim of this paper is to serve some basic results about this filtration. In this paper, we assume that $2-2g-n<0$, which is equivalent to that the group $\pi_{g,n}$ is non-abelian.

To explain our first result, let us recall that there exists a canonical exact sequence

$$1 \longrightarrow \pi_{g,n-1} \xrightarrow{d_*} \Gamma_{g,n} \xrightarrow{p_*} \Gamma_{g,n-1} \longrightarrow 1.$$

The homomorphism p_* is induced from "forgetting" the n-th puncture and the homomorphism d_* can be explicitly described by a result of Birman. Then, it can be shown that the homomorphisms d_* and p_* preserve the filtrations, hence we have a complex

$$(*) \qquad 0 \longrightarrow \mathrm{gr}(\pi_{g,n-1}) \longrightarrow \mathrm{gr}(\Gamma_{g,n}[1]) \longrightarrow \mathrm{gr}(\Gamma_{g,n-1}[1]) \longrightarrow 0.$$

Here, each associated graded module gr( ) has a structure of a graded Lie algebra. Moreover, if $g \geqq 1$, the Siegel modular group $\mathrm{Sp}(2g; \boldsymbol{Z})$ naturally acts on them.

THEOREM A. *If $n \geqq 2$, the complex (*) is an exact sequence of graded Lie algebras with* $\mathrm{Sp}(2g; \boldsymbol{Z})$*-action.*

Our second result is about the comparison of $\Gamma_{g,n}$ with the pro-l mapping class group, l being a fixed prime number. Using the pro-l completion $\pi_{g,n}^{\mathrm{pro}\text{-}l}$ of $\pi_{g,n}$ instead of $\pi_{g,n}$, we can define the pro-l mapping class group purely algebraically, which is denoted by $\Gamma_{g,n}^{(l)}$. (For the definition, see §4-1.) The group $\Gamma_{g,n}^{(l)}$ also has a filtration induced from the weight filtration of $\pi_{g,n}^{\mathrm{pro}\text{-}l}$.