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Metastable behaviors of diffusion processes with small parameter

By Makoto SUGIURA*)

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1. Introduction.

Let $\mathcal{M}$ be an orientable σ-compact d-dimensional Riemannian manifold of class C^{∞} with Riemannian metric $g=(g_{ij})$. Suppose a potential function $U\in C^{\infty}(\mathcal{M})$ is given and consider a second order differential operator $\mathcal{L}^{\varepsilon}$ on $\mathcal{M}$ defined by

$$\mathcal{L}^{\varepsilon}=\frac{\varepsilon^{2}}{2}\Delta-\frac{1}{2}\operatorname{grad}U, \qquad \varepsilon>0, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$ and grad means the Riemannian gradient. This paper is concerned with metastable behaviors of the diffusion process (x_t^{ε}, P_x) generated by $\mathcal{L}^{\varepsilon}$ on the space $\mathcal{M}$. Namely, we shall show that, for an appropriate choice of α_{ε}, the finite-dimensional distributions of the scaled process $\{x_{t\alpha_{\varepsilon}}^{\varepsilon}\}$ converge as $\varepsilon\downarrow 0$ to those of a Markov jump process living on the bottom $N=\{U=0\}$ of the potential. The results will be stated precisely in Section 2.

The metastable behaviors of diffusion processes have been studied by several authors, while all of them concern the diffusions on the Euclidean space with a double-well patential whose heights of the local minima are different from each other. For the one-dimensional Euclidean space, Kipnis and Newman [**10**] took up this problem and Ogura [**13**] solved it completely. Galves, Olivieri and Vares [**6**] considered the multi-dimensional case and used some smoothing and breaking procedure to obtain weak convergence on the path space, but the convergence of finite-dimensional distributions like this paper does not follow from their results.

There is a problem to be solved before establishing the metastable behaviors: namely, it should be determined the asymptotic behavior as $\varepsilon\downarrow 0$ of the first exit time

$$\tau_G^{\varepsilon}=\inf\{t>0\,;\,x_t^{\varepsilon}\notin G\} \tag{1.2}$$

of x_t^{ε} from a domain $G\subset\mathcal{M}$.

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