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A class of Riemannian manifolds with integrable geodesic flows

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Introduction.

The purpose of the present paper is to introduce a notion of geodesic flows—*simple integrability*. In a word, a simply integrable geodesic flow is a geodesic flow which can be integrated by a single quadratic function. A remarkable property of simple integrability is the duality: To a Riemannian manifold with simply integrable geodesic flow, there corresponds, through certain conformal change of the metric, such another Riemannian manifold. To be more precise, let $(M, \boldsymbol{g})$ be an n-dimensional Riemannian manifold $(n\geqq 2)$. For a tensor field ι of type $(1, 1)$ on M such that the determinant $\sigma_n(\iota)$ is positive on M, we introduce tensor fields ι_p of type $(1, 1)$ as follows:

$$\iota_p = \sigma_n(\iota)^{-(p-1)/(n-1)} \sum_{s=0}^{p-1} (-1)^s \sigma_s(\iota)\iota^{p-s}, \qquad p=1, \cdots, n.$$

Here we view ι as endomorphisms of tangent spaces, ι^{p-s} are the compositions iterated $p-s$ times, $\sigma_0(\iota)=1$, and $\sigma_s(\iota)$ denote the elementary symmetric polynomials of the eigenvalues of ι, of degree s.

DEFINITION. We say that the geodesic flow of $(M, \boldsymbol{g})$ is *simply integrable* if there exists a symmetric tensor field ι of type $(1, 1)$ with $\sigma_n(\iota)>0$ such that the n functions f_p on $T(M)$ defined by $f_p(X)=\boldsymbol{g}(\iota_p(X), X)$, $p=1, \cdots, n$, form a complete involutive set, i.e., are functionally independent and every Poisson bracket $\{f_p, f_q\}$ vanishes. We then call ι the *generating tensor field*. Note that simple integrability implies complete integrability in Liouville's sense, because $f_n(X)=(-1)^{n+1}\boldsymbol{g}(X, X)$.

The Riemannian manifolds with simply integrable geodesic flows have the following characteristic property.

MAIN THEOREM. *Suppose that the geodesic flow of $(M, \boldsymbol{g})$ is simply integrable with generating tensor field ι. Let $\tilde{\boldsymbol{g}}=\sigma_n(\iota)^{-1/(n-1)}\boldsymbol{g}$ be the conformal change of the metric. Then the geodesic flow of $(M, \tilde{\boldsymbol{g}})$ is simply integrable, and the generating tensor field is given by ι^{-1}.*