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Embedded flat tori in the unit 3-sphere

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1. Introduction.

Let S^3 be the unit hypersphere in the 4-dimensional Euclidean space E^4 given by $\sum_{i=1}^4 x_i^2=1$. For each θ with $0<\theta<\pi/2$, we consider a surface M_θ in S^3 defined by

$$x_1^2+x_2^2=\cos^2\theta\,, \qquad x_3^2+x_4^2=\sin^2\theta\,.$$

The surface M_θ, which is called a Clifford torus in S^3, can be viewed as an embedded flat torus in S^3. There are many other examples of embedded flat tori in S^3. Let $p: S^3\to S^2$ be the Hopf fibration, and let γ be a simple closed curve in S^2. Then it is known [4] that the inverse image $p^{-1}(\gamma)$ is an embedded flat torus in S^3. Note that $p^{-1}(\gamma)$ is foliated by great circles of S^3, and so it satisfies the *antipodal symmetry*, i.e., it is invariant under the antipodal map of S^3. Recently the author [2] obtained another example of embedded flat tori in S^3. Although this example contains no great circle of S^3, it also satisfies the antipodal symmetry. In this paper we show that the antipodal symmetry holds for all embedded flat tori in S^3. In other words, we prove the following theorem.

THEOREM 1.1. *If $f: M\to S^3$ is an isometric embedding of a flat torus M into S^3, then the image $f(M)$ is invariant under the antipodal map of S^3.*

REMARK. In Theorem 1.1 the word "embedding" cannot be replaced by "immersion". In fact, Theorem 4.4 says that there exists a flat torus M and an isometric immersion $f: M\to S^3$ such that the image $f(M)$ is not invariant under the antipodal map of S^3. However the author does not know the answer to the following question: *For every isometric immersion f of a flat torus M into S^3, does there exist a pair of points p and q in M such that $f(p)$ and $f(q)$ are antipodal points of S^3?*

The outline of this paper is as follows. Let $SU(2)$ be the group of all 2×2 unitary matrices with determinant 1. Then $SU(2)$, endowed with a bi-invariant metric, is isometric to S^3. Using the group structure on S^3, we define a