J. Math. Soc. Japan
Vol. 47, No. 1, 1995

# Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle

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(Received Sept. 1, 1992)
(Revised May 9, 1994)

Let $\mathfrak{X}(\boldsymbol{R}/\boldsymbol{Z})$ be a real vector space of bounded integrable functions on the circle which is invariant under the composition of any Lipschitz homeomorphism of the circle. That is, for any $\varphi\in\mathfrak{X}$ and any Lipschitz homeomorphism $f$ of the circle, $\varphi\circ f\in\mathfrak{X}$. Such a function space $\mathfrak{X}$ gives rise to a group $G^{L,\mathfrak{X}}(\boldsymbol{R}/\boldsymbol{Z})$ of Lipschitz homeomorphisms of the circle: an element of $G^{L,\mathfrak{X}}(\boldsymbol{R}/\boldsymbol{Z})$ is a Lipschitz homeomorphism $f$ of the circle such that $\log f'(x-0)$ belongs to $\mathfrak{X}$. The verification of the fact that $G^{L,\mathfrak{X}}(\boldsymbol{R}/\boldsymbol{Z})$ is a group is elementary ([**19**]).

The groups $G^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})=\mathrm{Diff}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ $(0<\alpha<1)$ are of course examples of such groups. This $G^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ is a subgroup of $G^{L,\mathcal{V}_{1/\alpha}}(\boldsymbol{R}/\boldsymbol{Z})$ defined in [**19**].

In order to define the group $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$, we need the notion of $\beta$-variation for a real number $\beta\geqq 1$. For a function $\varphi$ on $\boldsymbol{R}/\boldsymbol{Z}$ and a finite subset $A=\{x_1, \cdots, x_k\}$ of $\boldsymbol{R}/\boldsymbol{Z}$, we put

$$v_\beta(\varphi, A)=\sup\sum_{j=1}^{k}|\varphi(x_j)-\varphi(x_{j-1})|^\beta,$$

where $x_1, \cdots, x_k=x_0$ is in the cyclic order. Then we put

$$V_\beta(\varphi)=\sup v_\beta(\varphi, A),$$

where the supremum is taken over all finite subsets $A$ of $\boldsymbol{R}/\boldsymbol{Z}$. We call it the $\beta$-variation of $\varphi$. The functions on $\boldsymbol{R}/\boldsymbol{Z}$ whose $\beta$-variations are bounded form a linear space $\mathcal{V}_\beta(\boldsymbol{R}/\boldsymbol{Z})$ with the $\beta$-pseudonorm $|||\ |||_\beta$ defined by

$$|||\varphi|||_\beta=V_\beta(\varphi)^{1/\beta}.$$

We define $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$ to be the group of Lipschitz homeomorphisms $f$ with compact support such that $\log f'(x-0)$ exist as elements of $\mathcal{V}_\beta(\boldsymbol{R}/\boldsymbol{Z})$.

Then it is easy to see that $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$ contains both $G^{1+1/\beta}(\boldsymbol{R}/\boldsymbol{Z})$ and the group of class $P$ ([8]) which is denoted by $G^{L,\mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})$ in this paper. Note that $G^{L,\mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})$ contains the group $PL(\boldsymbol{R}/\boldsymbol{Z})$ of piecewise linear homeomorphisms whose homological property is rather well known by the work of Greenberg ([**7**]).