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# The versal deformation of the $E_6$-singularity and a family of cubic surfaces

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## 1. Introduction.

The purpose of this paper is to clarify the relation between the versal deformation of the $E_6$-singularity and a family of cubic surfaces originally due to A. Cayley.

We consider the cubic surface $S(p_0, p_1, p_2, q_0, q_1, q_2)$ defined by

$$x^3-2yz^2-y^2+x(p_0w^2+p_1zw+p_2z^2)+q_0w^3+q_1zw^2+q_2z^2w=0 \tag{1.1}$$

in $\boldsymbol{P}^3$ with homogeneous coordinate $(x:y:z:w)$, where $p_0, p_1, p_2, q_0, q_1, q_2$ are parameters. We frequently write $pq=(p_0, p_1, p_2, q_0, q_1, q_2)$ for simplicity. If we put $w=1$, the family of surfaces $S(pq)$ is regarded as the versal deformation of the rational double point of type $E_6$:

$$x^3-2yz^2-y^2=0$$

(cf. [Sl]). On the other hand, there is a long history on the study of cubic surfaces. Among others, we recall the 4-dimensional family of cubic surfaces due to A. Cayley (cf. [C]). Modifying his family, we introduce a family of cubic surfaces of $\boldsymbol{P}^3$ with homogeneous coordinate $(X:Y:Z:W)$ depending on parameters $(\lambda, \mu, \nu, \rho)$ as follows (cf. [NS]):

$$\begin{aligned}\rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2+(\mu\nu+1)YZ+(\lambda\nu+1)ZX \\ +(\lambda\mu+1)XY-(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]+XYZ=0.\end{aligned} \tag{1.2}$$

Since the moduli space of the cubic surfaces is 4-dimensional, the family above has enough parameters. For this reason, writing down the defining equation (1.1) in the form (1.2), we obtain a map $\Psi$: $pq\rightarrow(\lambda, \mu, \nu, \rho)$ at least in principle. Since the map $\Psi$ is multi-valued, we have to change the parameter space of $S(pq)$ to its covering space admitting a linear $W(E_6)$-action, where $W(E_6)$ is the Weyl group of type $E_6$, in order to define a single-valued map to the $(\lambda, \mu, \nu, \rho)$-space.

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