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Limit shape of the cross-section of shrinking doughnuts

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§1. Introduction.

In this article we are concerned with the asymptotic behaviour of symmetric 2-tori which are shrinking to a circle by the mean curvature flow.

The mean curvature flow problem, in a normal parametrization, is to find the family of hypersurfaces $F(M_0, t)=M_t \hookrightarrow \boldsymbol{R}^{n+1}$ $(n \geqq 2)$ satisfying

$$(1) \qquad \begin{cases} \dfrac{\partial F}{\partial t}(x, t) = -H(x, t)\cdot N(x, t) \\ F(x, 0) = F_0(x) : M_0 \hookrightarrow \boldsymbol{R}^{n+1}, \end{cases}$$

where N denotes the outward unit normal and H is the mean curvature with respect to N. Notice that in terms of the induced metric on M_t the right hand side of (1) is the Laplace-Beltrami operator Δ_{M_t} on M_t.

First we briefly recall some known facts on this problem. When the initial surface M_0 is strictly convex, Huisken [**18**], inspired by the work of Hamilton [**17**], showed that (1) shrinks M_0 to a round point within finite time, and also proved that for the area preserving rescaled flow M_0 really converges to a sphere in the C^∞-topology. Later Grayson [**15**] gave the counterexample which shows the convexity assumption in Huisken's theorem cannot be omitted; not all compact hypersurfaces with genus zero shrink to a point without singularity. Our previous joint work [**1**] with Ahara, on the other hand, dealt with the symmetric 2-torus and proved that under some technical hypothesis the torus might be shrunk to a circle by the mean curvature flow (see Theorem 2.1, below). Symmetry enables (1) to reduce essentialy a one-dimensional parabolic equation and hence our idea of the proof is based on applying the method of Gage and Hamilton [**11**], which discuss the plane curve shortening problem, to the equation for the generating curve. Although in our case there appears a lower order term in addition to the plane curve equation (see (6) below), our hypothesis in [**1**] makes it possible to apply the method of [**11**]. Indeed this

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