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A remark on the Kawamata rationality theorem

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Introduction.

Let X be a projective variety with Gorenstein, rational singularities. Let $\varphi: X\rightarrow Y$ be a surjective morphism with connected fibers from X to a normal projective variety Y. Let L be a φ-ample line bundle and assume that K_X is not φ-nef. Then the Kawamata rationality theorem states that there is a positive fraction $\tau=u/v$, where u, v are positive coprime integers, and such that

a) $K_X+\tau L$ is φ-nef but not φ-ample;

b) $u\leqq\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$.

If u takes on the maximal value, $\max_{y\in Y}\{\dim\varphi^{-1}(y)\}+1$, allowed by the Kawamata rationality theorem, then X is a $\boldsymbol{P}^{u-1}$ bundle over Y (see (2.2)). Moreover there is an ample line bundle $\mathcal{L}$ on X such that $K_X\otimes\mathcal{L}^u\approx\varphi^*H$ for an ample line bundle H on Y, and thus $X=\boldsymbol{P}(\mathcal{E})$ for the ample vector bundle $\mathcal{E}=\varphi_*\mathcal{L}$.

If L is ample and K_X is not nef, the Kawamata rationality theorem and the Kawamata-Shokurov base point free theorem imply that there is a fraction, $\tau=u/v$, with u, v positive coprime integers (called the *nef value* of the pair (X, L)) and a morphism $\phi: X\rightarrow Y$ with connected fibers (called the *nef value morphism* of the pair (X, L)) onto a normal projective variety Y such that

i) $vK_X+uL\approx\phi^*H$ for an ample line bundle H on Y,

ii) $u\leqq\max_{y\in Y}\{\dim\phi^{-1}(y)\}+1$.

We saw that $u=\max_{y\in Y}\{\dim\phi^{-1}(y)\}+1$ implies that $\phi: X\rightarrow Y$ is very special. In our main result, (1.4.2), we study the structure of the nef value morphism, ϕ, in the case when $u=\max_{y\in Y}\{\dim\phi^{-1}(y)\}$. If the nef value morphism is birational we need a smoothness assumption on X.

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