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Isomorphism of meromorphic function fields on Riemann surfaces

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1. Introduction.

It is known that two Riemann surfaces are conformally equivalent if and only if meromorphic function fields on them are $\boldsymbol{C}$-isomorphic. In this paper, we consider not $\boldsymbol{C}$-isomorphisms but field isomorphisms. A $\boldsymbol{C}$-isomorphism means that the restriction of the isomorphism to $\boldsymbol{C}$ is the identity mapping. We always assume that a field isomorphism maps the imaginary unit $\sqrt{-1}$ to itself. If two Riemann surfaces are conformally equivalent, then their meromorphic function fields are obviously field isomorphic. The converse was studied by Nakai-Sario [7]. Their main theorem is that if meromorphic function fields on two Riemann surfaces are field isomorphic, then the two Riemann surfaces are homeomorphic. As a consequence, if meromorphic function fields on two Riemann surfaces are field isomorphic, then the Riemann surfaces are simultaneously compact or noncompact. If the Riemann surfaces are noncompact, then their meromorphic function fields are $\boldsymbol{C}$-isomorphic, see Iss'sa [2] and Nakai-Sario [7]. Thus they are conformally equivalent. In contrast with non-compact case, if the Riemann surfaces are compact, then they do not need to be conformally equivalent. A counter example was shown by Heins [1]. Accordingly we shall be concerned with compact case.

For a Riemann surface R, let $M(R)$ denote the meromorphic function fields on R. We introduce an equivalence relation in the set of all Riemann surfaces. Let R and S be Riemann surfaces. We say that R and S are equivalent if and only if $M(R)$ is field isomorphic to $M(S)$. It is clear that this is an equivalence relation. Each equivalence class consists of all field isomorphic Riemann surfaces. For noncompact Riemann surfaces, each equivalence class consists of only one Riemann surface. We note that two conformally equivalent Riemann surfaces are regarded as the same surface. Let I be the set of compact Riemann surfaces R such that the equivalence class containing R consists of only R itself. Then $R \in I$ if $M(S)$ is field isomorphic to $M(R)$ implies that S is conformally equivalent to R. Let I_g be the set of all elements of I of genus g.