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Noncompact Liouville surfaces

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1. Introduction.

A (local) Liouville surface is by definition a surface with a Riemannian metric of the following form:

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

where $x=(x_1, x_2)$ is a local coordinate system, and f_i is a function of the single variable $x_i (i=1, 2)$. This type of metric has a special property. Define

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) are the canonical coordinates on the cotangent bundle. Then F is a first integral of the geodesic flow on the bundle, i.e., the Poisson bracket $\{F, E\}$ of F and the energy function

$$E=\frac{1}{2}\frac{1}{f_1(x_1)+f_2(x_2)}(\xi_1^2+\xi_2^2)$$

vanishes. As a matter of fact, the Liouville surfaces are characterized in terms of a first integral.

Let g be a Riemannian metric on a neighborhood U of a point $p\in\boldsymbol{R}^2$, and $E\in C^\infty(T^*U)$ the corresponding energy function. For a function $H\in C^\infty(T^*U)$ on the cotangent bundle, we denote by H_p the restriction of H to the cotangent space T_p^*U at p. The following proposition is classical.

PROPOSITION 1.1 ([**2**, Proposition 1.1]). *Assume that $F\in C^\infty(T^*U)$ satisfies the following conditions:*

(1) $\{F, E\}=0$,

(2) *F_q is a homogeneous polynomial of degree 2 for every $q\in U$,*

(3) $F_p\notin\boldsymbol{R}E_p$.

Then there is a coordinate system (x_1, x_2) on a (possibly smaller) neighborhood of p, and there are functions $f_i(x_i) (i=1, 2)$ such that