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Perturbation formula of eigenvalues in a singularly perturbed domain

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§1. Introduction.

In this paper, we will deal with the eigenvalues of Laplacian (Neumann B. C.) in a singularly perturbed domain and consider their elaborate characterization. The eigenvalue problem of Laplacian has been an important subject since many years ago and has been studied from the view point of physics, geometry, PDE and other fields of mathematics. In particular, the eigenvalue problem on singularly perturbed domains arise in several real phenomena of physcical situations. We study the Dumbbell shaped domain (cf. Fig. 1) which is related with sound phenomena of wind instruments and is also a simplest case of partial degeneration of domain. Beale [**4**] has first studied a spectral property of such domain. Actually he characterized the set of the eigenfrequencies and the scattering frequencies. Several related results on the eigenvalue problem have been obtained afterwards (see Hale and Vegas [**12**], Anné [**1**], Jimbo [**16**], Fang [**9**], Jimbo and Morita [**18**] and other papers in the references). With the aid of the results and methods in [**4**], we can easily see that the set of the eigenvalues (Neumann B. C.) of $\Omega(\zeta)$ (see Fig. 1) is divided into two parts (in some sense). One is associated with the fixed region $D=D_1\cup D_2$ and the other is with degenerating region $Q(\zeta)$. That is, the set of eigenvalue $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ can be expressed as follows,

$$\{\mu_k(\zeta)\}_{k=1}^{\infty}=\{\omega_k(\zeta)\}_{k=1}^{\infty}\cup\{\lambda_k(\zeta)\}_{k=1}^{\infty} \tag{1.1}$$

where $\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k$ and $\lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k$. Here $\{\omega_k\}_{k=1}^{\infty}$ is the set of eigenvalues of $-\Delta$ on D (Neumann B. C.) and $\{\lambda_k\}_{k=1}^{\infty}$ is that of $-d^2/dz^2$ on $L=\bigcap_{\zeta>0}Q(\zeta)$ with Dirichlet condition (see also [**1**], [**16**]). From $\omega_1=\omega_2=0$, $\omega_3>0$, $\lambda_1>0$ and (1.1), one can easily see that $\mu_2(\zeta)$ goes to 0 while $\mu_3(\zeta)$ is bounded away from 0 when $\zeta\to 0$. In [**9**], Fang obtained an elaborate convergence rate of $\mu_2(\zeta)$. In [**18**], we extended this result to more general cases and moreover we obtained some useful properties of the corresponding eigenfunctions which will be effectively used in this paper. In both papers [**9**] and [**18**], they dealt with only eigenvalues tending to 0, whose corresponding eigenfunctions are