J. Math. Soc. Japan
Vol. 44, No. 4, 1992

Some inequalities for minimal fibrations of surfaces of general type over curves

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(Received April 24, 1991)
(Revised Oct. 28, 1991)

Introduction.

Let $g: Y \to C$ be a surjective morphism from a smooth complex projective 3-fold onto a smooth curve. Assume that a generic fibre of g is an irreducible surface of general type. Then, composing divisorial contractions and flips, we can birationally modify Y into X, a normal, projective, $\boldsymbol{Q}$-factorial variety with only terminal singularities, in such a way that g induces a morphism $f: X \to C$, with K_X being f-nef [**Mo2**], [**Ka3**]. We call f a (relatively) minimal fibration of surfaces of general type over C. Since X is a $\boldsymbol{Q}$-factorial 3-fold, K_X^3 is a well-defined rational number which is independent of the choice of the relatively minimal model X. The aims of this article are (1) to estimate K_X^3 from below in terms of other geometric invariants and (2) to describe the structure of X when K_X^3 is small.

MAIN THEOREM 1. *Let $f: X \to C$ be a minimal fibration of surfaces of general type over C, a smooth projective curve of genus b. Let F be a general fibre of f.*

(1) *If $p_g(F) \geqq 3$ and $|K_F|$ is not composed of a pencil, then*

$$K_X^3 \geqq \frac{4(p_g(F)-2)}{p_g(F)}\left\{\frac{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+4\chi(\mathcal{O}_F)}{2(p_g(F)-2)}(b-1)-\chi(\mathcal{O}_X)\right\}$$

or equivalently,

$$b \leqq 1+\frac{p_g(F)\left\{K_X^3+\frac{4(p_g(F)-2)}{p_g(F)}\chi(\mathcal{O}_X)\right\}}{2\{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+4\chi(\mathcal{O}_F)\}}.$$

(2) *If $|K_F|$ is composed of a pencil and F is not a surface with $K_F^2=1$, $p_g(F)=2$, $q(F)=0$, then*

$$K_X^3 \geqq \frac{4(p_g(F)-1)}{p_g(F)}\left\{\frac{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+2\chi(\mathcal{O}_F)}{2(p_g(F)-1)}(b-1)-\chi(\mathcal{O}_X)\right\}$$

or equivalently,