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# The birational action of $\mathfrak{S}_5$ on $\boldsymbol{P}^2(\boldsymbol{C})$ and the icosahedron

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## § 1. Introduction.

The symmetric group $\mathfrak{S}_5$ on five letters 1, 2, 3, 4, 5 is generated by permutations $s_i=(i, i+1)$ $(i=1, 2, 3, 4)$. As is known, the 2-dimensional complex projective space $\boldsymbol{P}^2$ admits a birational action of $\mathfrak{S}_5$ in the following manner:

$$s_1:\ (\xi_1:\xi_2:\xi_3) \longrightarrow (\xi_1^{-1}:\xi_2^{-1}:\xi_3^{-1}),$$

$$s_2:\ (\xi_1:\xi_2:\xi_3) \longrightarrow (\xi_1:\xi_1-\xi_2:\xi_1-\xi_3),$$

$$s_3:\ (\xi_1:\xi_2:\xi_3) \longrightarrow (\xi_2:\xi_1:\xi_3),$$

$$s_4:\ (\xi_1:\xi_2:\xi_3) \longrightarrow (\xi_1:\xi_3:\xi_2).$$

Here $\xi=(\xi_1:\xi_2:\xi_3)$ means a homogeneous coordinate of $\boldsymbol{P}^2$. Putting $S=\{\xi\in\boldsymbol{P}^2;\ \xi_1\xi_2\xi_3(\xi_2-\xi_3)(\xi_3-\xi_1)(\xi_1-\xi_2)=0\}$, we find that each $s_i$ defines an automorphism of $\boldsymbol{P}^2-S$. Moreover, it is known that $\mathfrak{S}_5$ coincides with the group of birational actions $\varphi$ of $\boldsymbol{P}^2$ such that $\varphi|(\boldsymbol{P}^2-S)$ are automorphisms.

Suggested by a result of [S], N. Takayama showed that there are mutually disjoint twenty simply connected domains $D_i$ $(i=1, \cdots, 20)$ of $\boldsymbol{P}^2-S$ such that their union is open dense in $\boldsymbol{P}^2$ and that they are transitive by the $\mathfrak{S}_5$-action. On the other hand, it is well-known that the alternative group $\mathfrak{A}_5$ of the fifth degree is the symmetry group of the icosahedron which has twenty faces.

The purpose of the present paper is to give a description of the fundamental group $\pi_1(\boldsymbol{P}^2-S)$ in terms of combinatorial properties of the icosahedron. In particular, we shall introduce a group $B(\gamma)$ consisting of certain equivalence classes of sequences of the twenty simply connected domains and show that $\pi_1(\boldsymbol{P}^2-S)\cong B(\gamma)$. The precise statement is given in Theorem 8.11.

We are going to explain the contents of this paper briefly. In § 2, we shall define twenty simply connected domains of $\boldsymbol{P}^2-S$ and study their properties. In § 3, we shall construct the blowing up space $Z$ of $\boldsymbol{P}^2$ so that the proper transform $\tilde{S}$ of $S$ is the union of ten lines whose intersecting points are of