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Notes on the topology of folds

Dedicated to Professor Haruo Suzuki on his 60th birthday

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1. Introduction.

In this paper we study smooth maps $f: M^n \to N^p$ of n-manifolds into p-manifolds ($n \geqq p$) having only fold singular points and find some obstructions to the existence of such maps. In [15], Thom showed that, for generic maps $f: M^n \to \boldsymbol{R}^2 (n \geqq 2)$, the number of cusp singular points has the same parity as the euler number of M^n (see also [7]); in particular, there are no smooth maps $f: M^n \to \boldsymbol{R}^2$ having only fold singular points if the euler number of M^n is odd. Thom also showed that if $n-p+1$ is odd and the $(n-p+2)$-th Stiefel-Whitney class of M^n is non-zero, then there are no smooth maps $f: M^n \to \boldsymbol{R}^p$ having only fold singular points. Our main results of this paper are some generalizations of Thom's results.

In §3, we shall show the following.

THEOREM 1. *Let M^n be a closed manifold with odd euler number and N^p an even-dimensional manifold with $w_{p-1}(N^p)=0$ and $w_p(N^p)=0$ $(n \geqq p \geqq 2)$, where $w_i(N^p) \in H^i(N^p; \boldsymbol{Z}/2\boldsymbol{Z})$ denotes the i-th Stiefel-Whitney class of N^p. Then there exist no smooth maps $f: M^n \to N^p$ having only fold singular points.*

THEOREM 2. *Let N^p be a stably parallelizable manifold. Suppose that $n-p+1 (\geqq 1)$ is odd and that $w_i(M^n) \neq 0$ for some $i \geqq n-p+2$. Then there exist no smooth maps $f: M^n \to N^p$ having only fold singular points.*

Theorems 1 and 2 are consequences of a more general result (Proposition 3.2), which we shall prove by directly constructing a certain bundle map φ: $TM^n \oplus \varepsilon^1 \to TN^p$, where ε^1 is the trivial line bundle over M^n. Unfortunately, Theorems 1 and 2 do not hold if $n-p+1$ is even. In fact, we shall give an explicit example of a smooth map $f: M^4 \to \boldsymbol{R}^3$ with only fold singular points such that M^4 has odd euler number (Example 3.7). However, if we restrict ourselves to *simple* maps, we have the following.

THEOREM 3. *Let M^n be a closed orientable manifold with odd euler number and N^p an orientable manifold with $w_{p-1}(N^p)=0$ and $w_p(N^p)=0$ $(n \geqq p \geqq 2)$. Then*