J. Math. Soc. Japan
Vol. 44, No. 3, 1992

Strong minimal pair theorem for the honest polynomial degrees of Δ_2^0 low sets

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(Received Aug. 13, 1991)

§ 1. Introduction.

Homer [3] has shown that assuming P=NP there is a Δ_3^0 set which is minimal with respect to the honest polynomial time Turing reducibility, $\leqq_T^h$, while it is known that the honest polynomial time Turing degrees (hp-T degrees) of recursive sets are dense. In [3], Homer raised a question whether a recursively enumerable (r.e.) set can be $\leqq_T^h$ minimal. An affirmative answer has been given by Ambos-Spies [1] (assuming P=NP). He has shown that every high r.e. Turing degree contains a $\leqq_T^h$ minimal element. Downey [2], on the other hand, has proved that no low Turing degree contains a $\leqq_T^h$ minimal set. It has also been shown there that the hp-T degrees of low r.e. sets are dense. He asks if the hp-T degrees of Δ_2^0 sets are dense. An affirmative answer evidently implies P$\neq$NP. We notice that in contrast to the hp-T degrees, the polynomial time Turing degrees (p-T degrees) of all sets are dense, which can be proved by relativizing the proof of the density of the p-T degrees of recursive sets due to Ladner [4].

Concerning Downey's question, we shall prove the following strong minimal pair theorem which obviously implies the density of the hp-T degrees of Δ_2^0 low sets.

THEOREM. *If A and B are Δ_2^0 low sets such that $B<_T^h A$, then there are two sets C and D which satisfy the following two conditions:*

(1) $B <_T^h C <_T^h A$ *and* $B <_T^h D <_T^h A$,

(2) $\deg_T^h(B) = \deg_T^h(C) \wedge \deg_T^h(D)$.

In [5], Landweber, Lipton and Robertson have proved the strong minimal pair theorem for the p-T degrees of recursive sets. In § 2, we shall give a proof of the theorem for the hp-T degrees of recursive sets. The proof is a typical example of a Ladner style "looking back" technique. In § 3, we shall give a proof of the theorem for the Δ_2^0 low sets. Since our proof heavily depends on the notion of hp-T reducibility, it is not known whether the strong minimal pair theorem holds for the p-T degrees of Δ_2^0 low sets.