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Symmetric plane curves with nodes and cusps

Dedicated to Professor Heisuke Hironaka on his 60th birthday

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§ 1. Introduction.

In [**Z1**], Zariski considered the family of projective curves of degree 6 with 6 cusps on a conic. This family is defined by: $f(X, Y, Z)=f_2(X, Y, Z)^3+f_3(X, Y, Z)^2=0$ where f_i is a homogeneous polynomial of degree i, $i=2, 3$. He showed that the fundamental group $\pi_1(\boldsymbol{P}^2-C)$ is isomorphic to the free product $\boldsymbol{Z}_2*\boldsymbol{Z}_3$ for a generic member of this family. He also proved that the fundamental group of the complement of a curve of degree 6 with 6 cusps which are not on a conic is not isomorphic to $\boldsymbol{Z}_2*\boldsymbol{Z}_3$. In fact, we will show in § 5 that this fundamental group is abelian. Zariski also studied a curve of degree 4 with 3 cusps as a degeneration of the first family in [**Z1**] and he claims that the complement of such a curve has a non-commutative finite fundamental group of order 12. We will reprove this assertion (§ 3 Theorem (3.12)).

The purpose of this note is to construct systematically plane curves with nodes and cusps which are defined by symmetric polynomials $f(x, y)$. A symmetric polynomial $f(x, y)$ can be written as a polynomial $h(u, v)$ where $u=x+y$ and $v=xy$. In this expression, the degree of h in v is half of the original degree and the calculation of the fundamental group becomes comparatively easy. Let $p: \boldsymbol{C}^2\to\boldsymbol{C}^2$ be the two-fold branched covering defined by $p(x, y)=(u, v)$. The branching locus is the discriminant variety $D=\{u^2-4v=0\}$. Let $C=\{h(u, v)=0\}$ and $\tilde{C}=p^{-1}(C)$. Under a certain condition, the homomorphism $p_*: \pi_1(\boldsymbol{C}^2-\tilde{C})\to\pi_1(\boldsymbol{C}^2-C)$ is an isomorphism (Theorem (2.3), § 2). Symmetric polynomials give enough models for the cuspidal curves with small degree. In fact, we will give examples of symmetric plane curves of the following type and we will compute their fundamental groups.

(1) Symmetric curve of degree 4 with 3 cusps (Theorem (3.12), § 3).
(2) Symmetric curve of degree 5, with 4 cusps (Theorem (3.14), § 3).
(3) Symmetric curve of degree 6, with conical 6 cusps (Theorem (4.5), § 4).
(4) Symmetric curve of degree 6, with non-conical 6 cusps (Theorem (5.8), § 5).