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On regular subalgebras of Kac-Moody algebras and their associated invariant forms
—Symmetrizable case—

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Introduction.

In [**3**], Dynkin classified all the semi-simple subalgebras of finite dimensional complex semi-simple Lie algebras. There, a special kind of subalgebras with compatible root space decompositions, called *regular semi-simple subalgebras*, played an important role.

In this paper, we treat a Kac-Moody algebra with a symmetrizable generalized Cartan matrix (=GCM), and study its *regular subalgebras*, defined as a natural infinite dimensional analogue of Dynkin's ones. Though being no more isomorphic to Kac-Moody algebras in general, these regular subalgebras are isomorphic to generalized Kac-Moody algebras (=GKM algebras) introduced by Borcherds [**1**].

We now give a constructive definition of regular subalgebras. Let $\mathfrak{g}(A)$ be a Kac-Moody algebra with a symmetrizable GCM A, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(A)$, and Δ be the root system of $\mathfrak{g}(A)$. A subset $\{\beta_1, \cdots, \beta_m, \beta_{m+1}, \cdots, \beta_{m+k}\} \subset \Delta$ is called *fundamental* if it satisfies the following three conditions (see Definition 5.2):

(1) $\{\beta_r\}_{r=1}^{m+k} \subset \mathfrak{h}^*$ is a linearly independent subset;

(2) $\beta_s - \beta_t \notin \Delta \cup \{0\}$ $(1 \leqq s \neq t \leqq m+k)$;

(3) β_i is a real root $(1 \leqq i \leqq m)$ and β_j is a positive imaginary root $(m+1 \leqq j \leqq m+k)$.

Let $\tilde{\mathfrak{g}}$ be a subalgebra of $\mathfrak{g}(A)$ generated by root vectors attached to each root $\pm\beta_r$ $(1 \leqq r \leqq m+k)$ and a certain vector subspace $\mathfrak{h}_0$ of $\mathfrak{h}$. Then, $\tilde{\mathfrak{g}}$ is canonically isomorphic to a GKM algebra (see Theorem 5.1). We call this subalgebra $\tilde{\mathfrak{g}}$ a *regular subalgebra* of $\mathfrak{g}(A)$ after Dynkin.

The above definition of a fundamental subset and the construction of a subalgebra $\tilde{\mathfrak{g}}$ are generalizations of those by Morita in [8]. There, he considered only the case all β_r are real roots (i.e., $k=0$ in the above notation) and constructed a subalgebra $\hat{\mathfrak{g}}$, which coincides with the derived algebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ of the above $\tilde{\mathfrak{g}}$, in order to introduce certain subsystems of the root system Δ of