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On Noether's inequality for threefolds

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Introduction.

Let S be a complex minimal algebraic surface of general type. Let K_S be the canonical bundle of S and $p_g(S)$ be the geometric genus of S. Then in general, we have a classical inequality: $K_S^2 \geqq 2p_g(S)-4$, which is Noether's inequality.

In this paper, we will study a three-dimensional analogue. Since we have Noether's inequality for minimal surfaces (and also canonical models of surfaces) we expect some inequalities between the geometric genus and the cube of the first Chern class for three dimensional canonical models, which may be singular and not factorial. Very optimistically, we might expect that: for any canonical model X of a threefold of general type, we should have $K_X^3 \geqq 2p_g(X)-6$. But that is not the case in general.

MAIN THEOREM (Theorems 2.4, 3.1, 4.1). *Let X be a three-dimensional algebraic variety defined over $\boldsymbol{C}$. Assume that X has at most canonical singularities and that a canonical divisor K_X is nef and big. Let $d=\dim \Phi_{K_X}(X)$.*

(1) *If $d=3$, then $K_X^3 \geqq 2p_g-6$.*

(2) *If $d=2$ and K_X is Cartier, then either*

 2a) *$K_X^3 \geqq 2p_g(X)-4$ or*

 2b) *Φ_{K_X} is birationally equivalent to a fibration of curves of genus two with a rational section over a birationally ruled surface.*

(3) *If $d=1$, K_X is ample and X is factorial, then either*

 3a) *$K_X^3 \geqq 2p_g(X)-2$,*

 3b) *X is singular, the image is a rational curve, all the fibers are connected, $K_X^3=1$ and $p_g(X)=2$ or*

 3c) *the rational map Φ_{K_X} is a morphism and the general fibers of Φ_{K_X} are normal algebraic irreducible surfaces with only canonical singularities which have ample canonical divisors, $c_1^2=1$, $q=0$ and $1 \leqq p_g \leqq 2$.*

Moreover, the case 2b) really occurs. In (3.2), we will construct a smooth projective variety X with an ample canonical divisor K_X such that $\dim \Phi_{K_X}(X)$