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Motion of hypersurfaces and geometric equations

Dedicated to Professor Noboru Tanaka on his 60th birthday

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1. Introduction.

We are concerned with the motion of a hypersurface whose speed locally depends on the normal vector field and its derivatives. To be specific let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\boldsymbol{R}^n$ $(n \geqq 2)$ at time t. Let $\boldsymbol{n}$ denote the unit exterior normal vector field to $\Gamma_t = \partial D_t$. It is convenient to extend $\boldsymbol{n}$ to a vector field (still denoted by $\boldsymbol{n}$) on a tubular neighborhood of Γ_t such that $\boldsymbol{n}$ is constant in the normal direction of Γ_t. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The equation for Γ_t we consider here is of form

$$V = f(t, x, \boldsymbol{n}(x), \nabla \boldsymbol{n}(x)) \quad \text{on } \Gamma_t, \tag{1.1}$$

where f is a given function and ∇ stands for spatial derivatives. Material science provides a lot of examples of (1.1) where Γ_t is an interface bounding two phases of materials (see [**2**, **11**, **12**] and references therein). For example if

$$V = -\operatorname{div} \boldsymbol{n}, \tag{1.2}$$

the hypersurface Γ_t moves by its $(n-1$ times) mean curvature and (1.2) is known as the mean curvature flow equation. We note that this equation arises as a singular limit of some reaction-diffusion equations [**3**, **17**]. It is also important to consider anisotropic properties of materials. A typical model (cf. [**11**, **12**]) is

$$\beta(\boldsymbol{n})V = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\frac{\partial H}{\partial p_i}(\boldsymbol{n})\Big) + c, \tag{1.3}$$

where H is convex on $\boldsymbol{R}^n$ and positively homogeneous of degree one, β is a positive function on a unit sphere S^{n-1} in $\boldsymbol{R}^n$ and c is a constant. The equation (1.3) includes (1.2) as a particular example with $H(p) = |p|$, $\beta = 1$ and $c = 0$. We remark that in general the right hand side of (1.3) is *not* expressed as a