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On the order of growth of the Kloosterman zeta function

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§ 1. Introduction.

Let $H=\{z=x+iy\in \boldsymbol{C} \mid \operatorname{Im} z=y>0\}$ be the complex upper half plane given the Riemann structure

$$ds^2 = y^{-2}(dx^2+dy^2), \tag{1.1}$$

and let $G=PSL(2, \boldsymbol{R})=SL(2, \boldsymbol{R})/\{\pm 1\}$. Then the group G acts on H as linear fractional transformation:

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

and moreover the metric (1.1) gives rise to a G-invariant measure and the Laplace operator whose explicit forms are

$$d\mu(z) = y^{-2}dxdy \tag{1.2}$$

and

$$D = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{1.3}$$

Throughout this paper, we will suppose that $\Gamma(\subset G)$ is a congruence subgroup, though there is no need to make this restriction. In fact, all results given in this material can be generalized to any Fuchsian group of the first kind with a cusp ∞ by slight modifications. We further denote by $\mathscr{D}_\Gamma(=\Gamma\backslash H)$ the fundamental domain of Γ, which is always noncompact.

Let now $L^2(\mathscr{D}_\Gamma)$ be the Hilbert space consisting of all functions which are automorphic with respect to Γ and square integrable on $\mathscr{D}_\Gamma$, i.e.,

$$L^2(\mathscr{D}_\Gamma) = \left\{f \mid f(\gamma z) = f(z) \quad \text{for } \gamma\in\Gamma, \int_{\mathscr{D}_\Gamma} |f(z)|^2 d\mu(z) < \infty\right\}.$$

Then, the space $L^2(\mathscr{D}_\Gamma)$ has a spectral decomposition in accordance with the operation of D:

$$L^2(\mathscr{D}_\Gamma) = L_0^2(\mathscr{D}_\Gamma)\oplus \boldsymbol{C}\oplus L_c^2(\mathscr{D}_\Gamma),$$