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A geometric characterization of the groups M_{12}, He and Ru

By Richard WEISS*

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1. Introduction.

As in [1], we define a geometry $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_r; *)$ to be an ordered sequence of r pairwise disjoint non-empty sets $\mathcal{B}_i$ together with a symmetric incidence relation $*$ on their union $\mathcal{B}=\mathcal{B}_1\cup\cdots\cup\mathcal{B}_r$ such that if F is any maximal set of pairwise incident elements (i. e. a maximal flag), then $|F\cap\mathcal{B}_i|=1$ for $i=1, \cdots, r$. The number r is called the rank of Γ. The geometry Γ is called connected if the r-partite graph $(\mathcal{B}, *)$ is connected.

We recall that a generalized n-gon (for $n\geqq 2$) is a geometry $\Gamma=(\mathcal{P}, \mathcal{L}; *)$ of rank 2 such that the bipartite graph $(\mathcal{P}\cup\mathcal{L}, *)$ has diameter n and girth $2n$. The elements of $\mathcal{P}$ are called points and the elements of $\mathcal{L}$ lines. A generalized n-gon is called *thick* if every vertex of the graph $(\mathcal{P}\cup\mathcal{L}, *)$ has at least three neighbors. If $\Pi=(\mathcal{P}, \mathcal{L}; *)$ is a thick generalized n-gon, we define Π_0 to be the geometry $(\mathcal{F}, \mathcal{P}\cup\mathcal{L}; *)$, where $\mathcal{F}$ is the set of maximal flags of Π and $*$ the natural incidence relation. Then Π_0 is a generalized $2n$-gon having two lines through every point (but more than two points on a line). We will call such a generalized $2n$-gon *point-thin*.

The building attached to the group $PSp_4(p^k)$ is a generalized quadrangle. For $p=2$, this geometry, which we denote by $Q(k)$, is self-dual (see [3]), and $Q(k)_0$ is a point-thin generalized octagon on which $\mathrm{aut}(PSp_4(p^k))$ acts flag-transitively. Similarly, there is a self-dual generalized hexagon associated with the group $G_2(3^k)$, which we denote by $\mathcal{H}(k)$, such that $\mathrm{aut}(G_2(3^k))$ acts flag-transitively on the generalized dodecagon $\mathcal{H}(k)_0$. The building attached to the group ${}^2F_4(2^k)$ is a generalized octagon with $1+2^k$ points on a line. We call this octagon $\mathcal{O}(k)$ and write $\mathcal{O}(k)^\circ$ to denote its dual.

Let F be a non-maximal flag of a geometry $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_r; *)$. The set

$$J=\{i\,|\,\mathcal{B}_i\cap F\neq\emptyset\}$$

is called the type of F. For each $m\notin J$, let $\mathcal{B}_m^F=\{u\in\mathcal{B}_m\,|\,u*x \text{ for all } x\in F\}$.

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