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Symmetric solutions of the equation for the scalar curvature under conformal deformation of a Riemannian metric

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1. Introduction.

Let (M, g) be a compact n-dimensional Riemannian manifold without boundary ($n\geqq 3$) and S_g be its scalar curvature. Define a new metric $\hat{g}$ which is pointwise conformal to g by $\hat{g}:=u^{p-2}g$ with a positive smooth function u. Then its scalar curvature is given by

$$S_{\hat{g}}=u^{1-p}(a\Delta_g u+S_g u)$$

where $p=2n/(n-2)$, $a=4(n-1)/(n-2)$ and Δ_g is the Laplacian with respect to g, namely

$$\Delta_g u:=-g^{ij}\nabla_i\nabla_j u\,.$$

Now we are interested in the question of which kind of functions can be realized as scalar curvatures in the conformal class of g. In particular, we assume g is invariant under the action of some isometry group Γ, and consider Γ-invariant scalar curvatures realized with Γ-invariant metrics. From the above formula, f is realized if and only if there exists at least one solution of the following nonlinear elliptic differential equation:

$$(*)\qquad \begin{cases} a\Delta_g u+S_g u=fu^{p-1} \\ u>0 \end{cases} \quad \text{on } M.$$

In the case f is a constant, it is called the Yamabe problem, and if f has the same signature as the Yamabe invariant $\mu(M)$ (see Definitions 1.1), then $(*)$ has at least one solution (cf. [**17**], [**15**], [**1**], [**10**]; see also [8]). On the other hand, several authors discussed also about the case f is not a constant and it is known that in the case $(M, g)=(S^n, g_0)$ (that is the standard sphere), $(*)$ does not always have a solution (cf. [5], [**2**]) even if $f>0$ (namely f has the same signature as $\mu(S^n)>0$). Aubin [**1**] obtained the useful criterion for the existence of a solution of $(*)$, and several authors generalized it to the Γ-invariant case. From this criterion and using Green function, Schoen [**10**]