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Compact Liouville surfaces

Dedicated to Professor Noboru Tanaka on his 60th birthday

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Introduction.

A (local) Liouville surface is by definition a surface which is equipped with a riemannian metric of the following form:

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

where $x=(x_1, x_2)$ is a coordinate system, and f_i is a function of the single variable x_i $(i=1, 2)$. This type of metric is called a Liouville metric. A remarkable property of a Liouville surface is that the geodesic flow has the following first integral F. Let (x, ξ) be the canonical coordinate system on the cotangent bundle, and let

$$E=\frac{1}{2}\frac{1}{f_1(x_1)+f_2(x_2)}(\xi_1^2+\xi_2^2)$$

be the energy function associated with the riemannian metric g. If we put

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

then it is easy to see that the Poisson bracket $\{E, F\}$ vanishes. The ellipsoid is a classical example of the Liouville surface, which is originally due to Jacobi (see Darboux [3] and Klingenberg [7] on this example and historical remarks).

The main purpose of this paper is to give a proper definition of compact Liouville surfaces, and to classify them. It is classically known that a Liouville surface is locally characterized as a 2-dimensional riemannian manifold whose geodesic flow has a first integral which is a homogeneous polynomial of degree 2 on each fibre (cf. Darboux [3] Livre VI, Chapitre II). In §1 we first review this fact. This leads us to the following definition: A compact Liouville surface (S, g, F) is a compact 2-dimensional riemannian manifold (S, g) wnose geodesic flow has a first integral F which is fibrewise a homogeneous polynomial of degree 2, and which is not a constant multiple of the energy function E. We also assume that F does not come from a local Killing vector field, which means