J. Math. Soc. Japan
Vol. 43, No. 2, 1991

New properties of special varieties arising from adjunction theory

By Mauro C. BELTRAMETTI and Andrew J. SOMMESE

(Received Dec. 5, 1989)
(Revised June 25, 1990)

Introduction.

Let $X\hat{}$ be a connected projective submanifold of $\boldsymbol{P}_C$ and let $L\hat{}=\mathcal{O}_{\boldsymbol{P}_C}(1)_{X\hat{}}$. Studying the pair $(X\hat{}, L\hat{})$ adjunction theoretically leads to various classes of varieties with special fibre structures, e.g. scrolls and quadric fibrations. In this article we study these special fibre structures and show that they are in many cases even better behaved that might be expected. It is a blanket assumption in this paper that $\dim X\hat{}\geqq 3$.

The adjoint bundle, $K_{X\hat{}}+(n-1)L\hat{}$, is nef and big except for the following very special pairs (see [**S2**], [**S7**], [**SV**]):

a) $(X\hat{}, L\hat{})$ is either $(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$, a scroll over a curve, or a quadric Q in $\boldsymbol{P}^{n+1}$ with $L\hat{}_Q=\mathcal{O}_{\boldsymbol{P}^{n+1}}(1)_Q$,

b) $(X\hat{}, L\hat{})$ is a Del Pezzo variety, i.e. $K_{X\hat{}}\approx L\hat{}^{-(n-1)}$,

c) $(X\hat{}, L\hat{})$ is a quadric bundle over a smooth curve,

d) $(X\hat{}, L\hat{})$ is a scroll over a surface.

The definitions of scrolls and quadric bundles are given in (0.6).

Given such a pair with $K_{X\hat{}}+(n-1)L\hat{}$ nef and big, there exists a new pair (X, L), *the reduction* of $(X\hat{}, L\hat{})$, where X is smooth and L is ample, and

- ◆ there exists a morphism $\pi: X\hat{}\to X$ expressing $X\hat{}$ as X with a finite set B blown up, $L=(\pi_* L\hat{})^{**}$,
- ◆ $L\hat{}\approx\pi^* L-[\pi^{-1}(B)]$ (equivalently $K_{X\hat{}}+(n-1)L\hat{}=\pi^*(K_X+(n-1)L)$),
- ◆ $K_X+(n-1)L$ is ample (and in fact very ample by the main result of [**SV**]).

Throughout this introduction $(X\hat{}, L\hat{})$ will always be a pair as above with a reduction (X, L). It further follows that $K_X+(n-2)L$ is nef and big except for a small list of exceptional pairs (see [**S7**], [**Fj**]):

a) $(X, L)=(\boldsymbol{P}^4, \mathcal{O}_{\boldsymbol{P}^4}(2))$ or $(\boldsymbol{P}^3, \mathcal{O}_{\boldsymbol{P}^3}(3))$,

b) $(X, L)=(Q, \mathcal{O}_Q(2))$ where $(Q, \mathcal{O}_Q(1))$ is a quadric in $\boldsymbol{P}^4$,

c) there is a holomorphic surjection $\phi: X\to C$ onto a smooth curve, C, where $K_X^2\otimes L^3\approx\phi^* H$ for an ample line bundle H on C; in particular the general fibre of ϕ is $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$,