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Minimum index for subfactors and entropy. II

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Introduction.

V. Jones [22] constructed his celebrated theory on index for type II_1 factors by using the notion of coupling constant. Kosaki [25] extended Jones' index theory to that for conditional expectations between arbitrary factors based on Connes' spatial theory [9] and Haagerup's theory on operator valued weights [17]. For von Neumann algebras $M\supseteq N$, let $\mathcal{E}(M, N)$ denote the set of all faithful normal conditional expectations from M onto N, and $\mathcal{E}(M)$ the set of all faithful normal states on M. When $M\supseteq N$ is a pair of factor and subfactor with $\mathcal{E}(M, N)\neq\emptyset$, Kosaki's index Index E varies depending on $E\in\mathcal{E}(M, N)$. But it was shown in [18] (independently by Longo [27]) that if Index $E<\infty$ for some $E\in\mathcal{E}(M, N)$, then there exists a unique $E_0\in\mathcal{E}(M, N)$ which minimizes Index E for $E\in\mathcal{E}(M, N)$. So we can define the minimum index $[M:N]_0=$ Index E_0 for a pair $M\supseteq N$.

Starting with the von Neumann entropy, we have at present several kinds of entropies in noncommutative probability theory (see [3, 4, 10, 11, 12, 29, 41, 43] for instance). Pimsner and Popa [33] exactly estimated the entropy $H(M|N)$ of a type II_1 factor M relative to its subfactor N in terms of Jones' index. This entropy extends the conditional entropy in commutative probability theory, and was first used by Connes and Størmer [12] to study the Kolmogorov-Sinai entropy of automorphisms of finite von Neumann algebras. As the natural generalization of $H(M|N)$ for finite von Neumann algebras, Connes [10] defined the entropy $H_\varphi(M|N)$ for general von Neumann algebras $M\supseteq N$ and a normal state φ on M by using the notion of relative entropy. Here the relative entropy of normal positive functionals was first studied by Umegaki [41] in the semi-finite case, and was extended by Araki [3, 4] to the general case. On the other hand, taking account of Pimsner and Popa's estimate of $H(M|N)$, we introduced in [19] another entropy $K_\varphi(M|N)$ of a von Neumann algebra M relative to its subalgebra N and $\varphi\in\mathcal{E}(M)$ such that $E\in\mathcal{E}(M, N)$ with $\varphi\circ E=\varphi$ exists. For factors $M\supseteq N$ and $E\in\mathcal{E}(M, N)$, we write $K_E(M|N)$ for $K_\varphi(M|N)$

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