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Fourier transforms for affine automorphism groups on Siegel domains

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Introduction.

Let G be a connected Lie group, dg a left Haar measure on G and π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. Then for an integrable function φ on G the Fourier transform with respect to π is defined as the integrated operator $\pi(\varphi)=\int_G \pi(g)\varphi(g)dg$. As is well known, if G is semi-simple or nilpotent, then $\pi(\varphi)$ is a compact operator on $\mathcal{H}$ for any irreducible representation π and any integrable function φ. But otherwise, $\pi(\varphi)$ is not always compact, thus characterization of $\varphi\in L^1(G)$ such that $\pi(\varphi)$ is compact is an important problem in representation theory for solvable Lie groups.

In [7], Khalil determined such functions for the $ax+b$ group by the "mean value over the subgroup of translations" (Example 3.1). In this paper we generalize this result to transitive groups of affine automorphisms on Siegel domains. More precisely, we treat connected and simply connected Lie groups G whose Lie algebras $\mathfrak{g}$ are normal j-algebras (Definition 1.1) and their square integrable representations.

Our characterization is, roughly speaking, based on conditions of zero-sets of partial Euclidean Fourier transform on the abelian normal subgroup $G_1=\exp\mathfrak{g}_1$ (under the notations of 1.5). Identifying G with $\mathfrak{g}_1\times(G_1\backslash G)$, we take the Euclidean Fourier transform $\mathcal{F}_1\varphi$ of $\varphi\in L^1(G)$ on $\mathfrak{g}_1$-part, which is a function on $\mathfrak{g}_1^*\times(G_1\backslash G)$. On the other hand, the unitary dual $\hat{G}$ of G being parametrized by coadjoint orbits of G on $\mathfrak{g}^*$, square integrable representations correspond to open orbits, whose union is dense in $\mathfrak{g}^*$. For such a representation π of G, let Ω be the corresponding open orbit, $\partial\Omega$ be its boundary in $\mathfrak{g}^*$. Considering the natural projection $p:\mathfrak{g}^*\to\mathfrak{g}_1^*$ defined by $p(l)=l|_{\mathfrak{g}_1}$ (restriction of l to $\mathfrak{g}_1$), we show that $\pi(\varphi)$ is compact if and only if $\mathcal{F}_1\varphi$ vanishes on $p(\partial\Omega)\times(G_1\backslash G)$ (Theorem 2.2).

In section 1, we summarize preliminary results on structures of normal j-algebras and unitary representations of their corresponding groups. Our