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Examples of degenerations of Castelnuovo surfaces

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Introduction.

Let $\rho : \mathcal{S} \to \Delta_\varepsilon$ be a proper flat morphism of a nonsingular threefold $\mathcal{S}$ to a disk $\Delta_\varepsilon = \{t \in \boldsymbol{C} ; |t| < \varepsilon\}$ with connected fibers. We call it a semi-stable degeneration if ρ is smooth over $\Delta_\varepsilon^* = \Delta_\varepsilon \setminus \{0\}$ and $S_0 = \rho^{-1}(0)$ is a reduced divisor with simple normal crossings. The divisor $S_t = \rho^{-1}(t)$ is called the singular fiber if $t=0$ and it is called a general fiber if $t \neq 0$. Let $\rho : \mathcal{S} \to \Delta_\varepsilon$ be a semi-stable degeneration and denote by $\rho^\circ : \mathcal{S}^* \to \Delta_\varepsilon^*$ its restriction to the punctured disk. Then the fundamental group of Δ_ε^* acts naturally on $H^2(S_t, \boldsymbol{Z})$, $t \neq 0$. Let N denote the logarithm of the monodromy action. We say the degeneration $\rho : \mathcal{S} \to \Delta_\varepsilon$ is a Type I (resp. Type II) degeneration if $N=0$ (resp. $N^2=0$). Type I degenerations are attractive, since one must study them if he want to make the period mapping proper.

In this article, we construct some semi-stable degenerations such that a general fiber is a *Castelnuovo surface*, that is, a minimal algebraic surface of general type with $c_1^2 = 3p_g - 7$ whose canonical map is birational onto its image.

In § 1, we recall some fundamental results on Castelnuovo surfaces found in [**1**]. In § 2, we construct a Type I degeneration. Note that a Castelnuovo surface with $p_g = 4$ is a quintic surface. Therefore, ours serves an explicit example of Type I degenerations of quintic surfaces whose existence was shown by Friedman [**4**] using Horikawa's family of deformations of a numerical quintic surface of type II_b [**5**]. We also refer the reader to [**4**] for further discussions on such degenerations. Friedman informed us that N. Shepherd-Barron constructed another Type I degeneration of quintic surfaces. The other examples of Type I degenerations of surfaces of general type can be found in [**4**], [**11**] and [**12**].

In § 3, we extend Horikawa's canonical resolution of singularities on double coverings of surfaces [**5**] to the case of cyclic triple coverings. This is used in § 4 in order to construct Type II degenerations. In our example, the singular fiber consists of a Castelnuovo surface Σ and a rational surface R, and the invariants of Σ are the "next" to those of a general fiber S_t on the line $c_1^2 = 3p_g - 7$, that is, $p_g(\Sigma) = p_g(S_t) - 1$ and $c_1^2(\Sigma) = c_1^2(S_t) - 3$. Thus we can descend