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The q-bracket product and quantum enveloping algebras of classical types

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1. Introduction.

Let A be an algebra over a commutative ring R and let $x_1, \cdots, x_n \in A$. We say A is a *polynomial algebra in* $x_1, \cdots, x_n$ if the set of all monomials $x_1{}^{a_1} \cdots x_n{}^{a_n}$, $a_i \in \boldsymbol{N}$, forms a free base for the R-module A. Note that the concept depends on the total ordering on generators.

If L is a finite dimensional Lie algebra over a field k with a base $z_1, \cdots, z_N$, the famous Poincaré-Birkhoff-Witt theorem tells that the universal enveloping algebra $U(L)$ is a polynomial algebra in $z_1, \cdots, z_N$.

Let $(a_{ij})_{1\leqq i, j\leqq n}$ be a symmetrizable generalized Cartan matrix. The corresponding quantum enveloping algebra $\hat{U}$ was introduced by Drinfeld [**2**, **3**] and Jimbo [**4**]. We follow Lusztig's formulation [**6**].

Take integers $d_i \neq 0$ such that $d_i a_{ij} = d_j a_{ji}$. Let k be a field with $q \in k^{\times}$ such that $q^{4d_i} \neq 1$ $(1 \leqq i \leqq n)$. $\hat{U}$ is the k-algebra (associative with 1) with generators e_i, f_i, k_i, k_i^{-1} $(1 \leqq i \leqq n)$ and relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i, \tag{1.1}$$

$$k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j, \qquad k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j, \tag{1.2}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q^{2d_i} - q^{-2d_i}}, \tag{1.3}$$

$$\sum_{\nu=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} e^{1-a_{ij}-\nu} e_j (-e_i)^{\nu} = 0 \qquad (i \neq j), \tag{1.4}$$

$$\sum_{\nu=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} f_i^{1-a_{ij}-\nu} f_j (-f_i)^{\nu} = 0 \qquad (i \neq j). \tag{1.5}$$

Here, we use the notations

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t}{[n]_t [m-n]_t} \in \boldsymbol{Z}[t, t^{-1}],$$

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