J. Math. Soc. Japan
Vol. 42, No. 2, 1990

Determination of the modulus of quadrilaterals by finite element methods

By Heihachiro HARA and Hisao MIZUMOTO

(Received March 28, 1989)

Introduction.

In the present paper we aim to establish a method of finite element approximations by which we can determine the modulus of quadrilaterals on Riemann surfaces (cf. Mizumoto and Hara [**15**] for other treatment). Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation not only for the case of Riemann surfaces but also for the case of plane domains.

Let Ω be a simply connected subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W. We assume that the boundary $\partial\Omega$ of Ω is a piecewise analytic curve. We assign four points p_1, p_2, p_3 and p_4 on $\partial\Omega$ (in positive orientation w.r.t. Ω), and the two opposite arcs C_0 (from p_1 to p_2) and C_1 (from p_3 to p_4). Then we say that a *quadrilateral* Q with opposite sides C_0 and C_1 is given.

We can conformally map the domain Ω onto a rectangular domain $R=\{w \mid 0<\operatorname{Re} w<1, 0<\operatorname{Im} w<M\}$ by a function $w=\mathfrak{f}(p)$ so that p_1, p_2, p_3 and p_4 are mapped to iM, 0, 1 and $1+iM$ respectively. Let $\mathfrak{F}$ be the class of all continuous functions v on $\bar{\Omega}$ with $v=0$ on C_0 and $v=1$ on C_1 which satisfy some restricted conditions (see §2.1). Then the modulus $M(Q)=M$ of the quadrilateral Q is uniquely determined by Q, and is given by

$$M(Q)=D(u)=\min_{v\in\mathfrak{F}} D(v) \qquad (u=\operatorname{Re}\mathfrak{f}(p)),$$

where by $D(v)$ we denote the Dirichlet integral of v. Next we assign the two opposite arcs $\tilde{C}_0$ (from p_2 to p_3) and $\tilde{C}_1$ (from p_4 to p_1) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with the opposite sides $\tilde{C}_0$ and $\tilde{C}_1$ is defined. We can easily see that $M(Q)=1/M(\tilde{Q})$. By making use of this relation Gaier [**9**] presented a method to obtain upper and lower bounds for the modulus $M(Q)$ in the case of some restricted domain Ω (e.g. a lattice domain, etc.) by the finite difference and

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60540110), Ministry of Education, Science and Culture, Japan.