J. Math. Soc. Japan
Vol. 42, No. 2, 1990

Metric deformation of non-positively curved manifolds

By Koji FUJIWARA

(Received Sept. 29, 1988)
(Revised Feb. 14, 1989)

§1. Introduction and statement of results.

There are strong relations between the topology and the curvature of a Riemannian manifold. For example, let M be a compact Riemannian manifold of negative curvature. Then every abelian subgroup of $\pi_1(M)$ must be cyclic, which is not necessarily true for a manifold of non-positive curvature.

A natural question is under what conditions a metric of non-positive curvature can be deformed to a metric of negative curvature. For this question, we have the following results.

THEOREM 1. *Let (M, g) be a complete Riemannian manifold with $K_g \leqq 0$, where K_g denotes the sectional curvature of (M, g), and p a point in M. Then there is a positive number R which is determined by the metric g and its derivatives around p, such that the following holds; suppose $K_g<0$ on $M \setminus B_R(p)$, then there is a metric $\bar{g}$ such that $K_{\bar{g}}<0$ and $g=\bar{g}$ on $M \setminus B_R(p)$, where we put $B_R(p)=\{q \in M;\ d(p, q)<R\}$.*

In general, the number R in Theorem 1 is much smaller than $i(p)$, the injectivity radius at p, but for two dimensional manifolds, we have a better result.

THEOREM 2. *Let (M, g) be a complete Riemannian manifold of two dimension with $K_g \leqq 0$. Suppose there is a point p in M such that $K_g<0$ on $M \setminus B_{i(p)}(p)$. Then there is a complete metric $\bar{g}$ such that $K_{\bar{g}}<0$ and $g=\bar{g}$ on $M \setminus B_{i(p)}(p)$.*

As a corollary to Theorem 2, we have the following result for $\boldsymbol{R}^2$.

COROLLARY OF THEOREM 2. *Let $(\boldsymbol{R}^2, g)$ be a complete metric on $\boldsymbol{R}^2$ with $K_g \leqq 0$. Suppose there is a compact set $A \subset \boldsymbol{R}^2$ with $K_g<0$ on $\boldsymbol{R}^2 \setminus A$. Then there is a complete metric $\bar{g}$ on $\boldsymbol{R}^2$ with $K_{\bar{g}}<0$ and $g=\bar{g}$ on $\boldsymbol{R}^2 \setminus B$ for some compact set $B \subset \boldsymbol{R}^2$.*

Generally, it is not possible to change a metric of non-positive curvature to a metric of negative curvature, because there is a topological obstruction between them as is stated before. But if the set of points at which K_g takes the zero is contained in a topologically trivial ball, then it is likely that we can