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On the unit groups of Burnside rings

Dedicated to the memory of Professor Akira Hattori

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1. Introduction.

Let G be a finite group. The set $A^{+}(G)$ of the G-isomorphism classes of finite right G-sets makes a commutative semi-ring with respect to disjoint union $+$ and Cartesian product $\times$. Its Grothendieck ring is called the *Burnside ring* of G and is denoted by $A(G)$. A finite (right) G-set is the disjoint union of its orbits and each orbit is G-isomorphic to a homogeneous G-set $H\backslash G:=\{Hg \mid g\in G\}$. Two G-sets $H\backslash G$ and $K\backslash G$ are isomorphic if and only if $H=_{G}K$, that is, H is G-conjugate to K. Thus this ring is additively a free abelian group on $\{[H\backslash G] \mid (H)\in Cl(G)\}$, where $Cl(G)$ is the conjugacy classes (H) of subgroups H of G.

A *super class function* is a map of the set of subgroups of G to $\boldsymbol{Z}$ which is constant on each conjugacy class of subgroups. Let $\tilde{A}(G):=\boldsymbol{Z}^{Cl(G)}$ be the ring of integral valued super class functions. For any subgroup S of G, the map $[X]\mapsto|X^{S}|$, the number of fixed-points, extends to a ring homomorphism φ_{S}: $A(G)\rightarrow\boldsymbol{Z}$, and so we have a ring homomorphism

$$(1)\qquad \varphi:=\prod_{(S)}\varphi_{S}:\ A(G)\longrightarrow\tilde{A}(G):=\boldsymbol{Z}^{Cl(G)};\quad [X]\longmapsto(|X^{S}|).$$

It is well-known that this map is injective. Thus we can identify any element x of $A(G)$ with the super class function $\varphi(x)$, and so we simply write

$$x(S):=\varphi(x)(S)=\varphi_{S}(x)$$

for a subgroup S of G. Hence we can view the unit group $A(G)^{*}$ as a subgroup of $\{\pm 1\}^{Cl(G)}$.

Now, tom Dieck proved by a geometric method that for any $\boldsymbol{R}G$-module V the function

$$u(V):\quad S\longmapsto \operatorname{sgn}\dim V^{S}$$

belongs to the Burnside ring $A(G)$, where $\operatorname{sgn} m:=(-1)^{m}$ ([**Di79**, Proposition 5.5.9]). The first purpose of this paper is to prove this fact by a purely alge-