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Periodic behavior of solutions to parabolic-elliptic free boundary problems

Dedicated to Professor Niro Yanagihara on his 60th birthday

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1. Introduction.

In the papers [**14**, **15**, **16**], parabolic-elliptic variational inequalities, formulated in a domain $\Omega\subset\boldsymbol{R}^N$ ($N\geqq 1$), with some time-dependent constraints have been considered. The existence and uniqueness theorems were there established with some results on asymptotic stability of solutions. In some cases the constraint is an obstacle imposed at a time-dependent part of the boundary Γ of Ω. We deal here with a mathematical model of a parabolic-elliptic free boundary problem which arises in the flow with saturation and unsaturation in porous media, when the water level of the reservoir changes periodically in time. For related papers concerning the analysis of saturated-unsaturated flows in porous media we refer to [**1**, **2**, **3**, **5**, **6**, **7**, **8**, **9**, **10**, **11**, **17**, **18**].

The problem is formulated as follows. Assume that for each $t\in\boldsymbol{R}$, Γ consists of disjoint three parts $\Gamma_D(t)$, $\Gamma_N(t)$ and $\Gamma_U(t)$, i.e., $\Gamma=\Gamma_D(t)\cup\Gamma_N(t)\cup\Gamma_U(t)$ (disjoint union). Correspondingly, for a given interval $J=(t_0, \infty)$ we define non-cylindrical subsets $\Sigma_\nu(J)$, $\nu=D, N, U$, of $\Sigma(J)=J\times\Gamma$ by

$$\Sigma_\nu(J)=\bigcup_{t\in J}\{t\}\times\Gamma_\nu(t), \qquad \nu=D, N, U.$$

The problem is described in the following system:

$$\rho(v)_t-\Delta v=f \qquad \text{in } Q(J)=J\times\Omega, \tag{1.1}$$

$$v=g_D \qquad \text{on } \Sigma_D(J), \tag{1.2a}$$

$$\partial_n v=q_N \qquad \text{on } \Sigma_N(J), \tag{1.2b}$$

$$v\leqq g_U,\quad \partial_n v\leqq q_U,\quad (\partial_n v-q_U)(v-g_U)=0 \qquad \text{on } \Sigma_U(J) \tag{1.2c}$$

and the initial condition

$$\rho(v(t_0, \cdot))=u_0 \qquad \text{for } x\in\Omega,$$