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On the stability of Riemannian manifolds

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§ 0. Introduction.

A map $f:(M, g)\rightarrow(N, h)$ from a compact Riemannian manifold (M, g) into a complete Riemannian manifold (N, h) is *harmonic* if it is a critical point for the energy integral $E(f)=\int_{M}|df|^{2}dv_{g}$.

The identity map of a compact Riemannian manifold is always a harmonic map. Any harmonic map has its Jacobi operator determined by the second variational formula of the energy integral of the harmonic map. The Jacobi operator of the identity map of a compact manifold is a linear elliptic self-adjoint operator of second order on the vector fields of the manifold. So we consider the first eigenvalue of the Jacobi operator of the identity map. We call a Riemannian manifold *stable* if the first eigenvalue of the Jacobi operator of the identity map is non-negative and *unstable* otherwise.

The stability of Riemannian manifolds has been studied by many people. Mostly they studied which Riemannian manifolds are stable or unstable. We consider the stability problem from the different point of view. We are interested in the problem how stability of a compact Riemannian manifold depends on its Riemannian metric. For example the three-dimensional sphere is unstable with its standard Riemannian metric but there also exists a Riemannian metric which makes the three-sphere stable. We formulate the problem as follows:

We consider all the possible Riemannian metrics on a given compact manifold. Then what are the possible signs of the first eigenvalues of Jacobi operators of the identity map of the manifold?

In this paper we give the complete answer to the problem if the dimension of the manifold is less than or equal to three and a partial result if the dimension of the manifold is greater than three.

The essential part of the proof is that we can construct an unstable Riemannian metric on the Euclidean ball of dimension greater than or equal to three.

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