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A stochastic approach to a Liouville property for plurisubharmonic functions

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1. Introduction.

It is well known that any plurisubharmonic function u on C^n increasing slower than logarithmic order at infinity must be constant. This kind of the Liouville property of the plurisubharmonic functions has been extended to certain complex manifolds. The purpose of this paper is to prove assertions of this type by making use of weakly recurrent holomorphic diffusion processes associated with plurisubharmonic exhaustion functions on the complex manifolds. As an application, we give a refinement of a recent result due to Takegoshi [**16**] concerning Kähler manifolds with poles.

In the case of the complex plane C any subharmonic function u defined on C does not exceed the harmonic function $v_R(z)=\sup_{|w|\leq 1}u(w)(1-\log|z|/\log R)+m(u, R)\log|z|/\log R$ on $\{1<|z|<R\}$, where $m(u, R)=\sup_{|z|\leq R}u(z)$. Letting $R\rightarrow\infty$, we have $\sup_{|w|\leq 1}u(w)\geqq u(z)$ over C provided that $\lim_{R\rightarrow\infty}m(u, R)/\log R=0$. Therefore u becomes constant by the maximum principle. We observe in this argument that $\log|z|/\log R$ is just the probability that the standard complex Brownian motion exits from the circle $\{|z|=R\}$ before hitting the inner circle $\{|z|=1\}$. This observation suggests a probabilistic method to work with plurisubharmonic functions on complex manifolds which admit holomorphic diffusions $M=\{Z_t, \zeta, \mathcal{F}_t, P_z\}$ enjoying a kind of recurrence property. In fact any plurisubharmonic function becomes M-subharmonic in the sense of Dynkin [**4**] for every holomorphic diffusion M. The term "holomorphic" is due to the property of M that the composite $f(Z_t)$ of the sample path Z_t with any holomorphic function f is a local martingale.

In § 2, we present a general property of subharmonic functions with respect to a weakly recurrent diffusion, generalizing the above mentioned argument for the Brownian motion. It will then be shown in § 3 that a Liouville type theorem for plurisubharmonic functions holds on a complex manifold of dimension n possessing a plurisubharmonic exhaustion function Ψ such that $(dd^c\Psi)^n$ tends to zero in a certain sense as $\Psi\rightarrow\infty$ (Theorem 1). By making use of this exhaustion function Ψ, we can construct a Dirichlet form following Fukushima-