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On a question raised by Conway-Norton

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0. Introduction.

Let G be a finite group and F be the collection of all modular functions $f(z)$ satisfying:

(1) $f(z)$ is a modular function with respect to a discrete subgroup Γ of $SL_2(\boldsymbol{R})$ of the first kind. (i.e. $f(z)$ is meromorphic on $H^*=H\cup\{\text{cusps of } \Gamma\}$ where H is the upper half plane.)

(2) The genus of Γ is zero and $f(z)$ is a generator of a function field of Γ (i.e. the genus of $\Gamma\backslash H^*$ is zero and $f(z)$ is a generator of a function field of $\Gamma\backslash H^*$).

(3) At $z=i\infty$, $f(z)$ has a Fourier expansion of the form:

$$q^{-1}+a_0+\sum_{n=1}^{\infty}a_nq^n \qquad (q=e^{2\pi iz}).$$

In [2], Conway and Norton have assigned a "Thompson series" of the form:

$$T_\sigma=q^{-1}+H_1(\sigma)q+H_2(\sigma)q^2+\cdots\in F$$

to each element σ of the Fischer-Griess "Monster" group M and conjectured that H_n are characters of M for all n. This remarkable connection between the "Monster" M and modular functions is called *Monstrous Moonshine*.

One of the problem which arose from Conway-Norton paper is that

(*) *For each element σ in $\cdot 0$, is there a class of elements σ_1 in M whose Thompson series T_{σ_1} has a form $\Theta_\sigma(z)/\eta_\sigma(z)+$constant?* (For the definition of $\eta_\sigma(z)$ and $\Theta_\sigma(z)$ see (1.3) and (1.4).)

In [2], Conway and Norton studied elements in $\cdot 0$ of weight 0 and proved that (*) is true for elements of weight 0 (i.e. if σ is of weight 0, then there is a class of elements σ_1 in M whose Thompson series T_{σ_1} has a form $\Theta_\sigma(z)/\eta_\sigma(z)$ +constant). In [6], Kondo and Tasaka studied elements in M_{24} (M_{24} can be naturally embedded in $\cdot 0$) and proved that (*) is true for elements in M_{24}. Recently, Kondo [8] calculated $\Theta_\sigma(z)$ for σ in $2^{12}M_{24}\backslash M_{24}$ and proved that (*)