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On the rationality of complex homology 2-cells: II

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§ 8. Introduction.

The purpose of this paper is to prove:

8.1. THEOREM. *Let X be an irreducible, smooth projective surface/$\boldsymbol{C}$, with the geometric genus $p_g(X)=0$. Let D be a reduced (not necessarily connected) curve on X with at worst ordinary double point singularities. Suppose each connected component of* Supp D *is simply connected and the irreducible components of D generate the divisor class group* Pic(X). *Then X is rational.*

In [**G-S**] we showed that such a surface X cannot be a surface of general type. Thus 8.1 follows from:

8.2. THEOREM. *Let X and D be as in* 8.1. *Assume further that X is an elliptic surface. Then X is rational.*

We will refer to the paper [**G-S**] by Part I. As already indicated in Part I, one easily deduces from 8.1, that any complex homology 2-cell is rational, thus answering affirmatively a question of Van de Ven. (For other consequences of 8.1 see Part I.) The reader is assumed to be familiar with Part I of this paper, the notations and conventions of which we continue to use here also. We shall now briefly outline the proof of 8.2 here.

8.3. Recall that (in Part I), we begin with the assumption that X is not rational (or, equivalently, that $|nK|\neq\emptyset$ for some $n>0$) so that $K+D$ has Zariski decomposition: $K+D=P+N$. Without loss of generality we also assume that there are no (-1)-curves E on X such that (i) $E\cdot D=1$ or (ii) $E\cdot D=2$ and E meets two different connected components of D. We then apply Miyaoka's inequality to (X, D). Then, by studying the blowing-down process $\pi: X\rightarrow X''$, where X'' is the minimal model for the function field of X, we obtain the auxiliary inequality

$$3(b_2-\beta_2)+b_0+\lambda+\sigma+\tau+e_1+r_3+2r_4 \leqq \beta_2''-5 \tag{2.8'}$$

where each term on the left hand side is nonnegative.