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Rate of decay at high energy of local spectral projections associated with Schrödinger operators

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§1. Introduction.

Let $\tilde{\mathfrak{H}}=-(1/2)\Delta+V(x)$, $\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$, be a Schrödinger operator on $\boldsymbol{R}^n$, $n\geqq 1$. We assume that the potential $V(x)$ satisfies the following assumption for some $m\geqq 0$.

ASSUMPTION $(\mathrm{A})_m$. $V(x)$ is a real-valued C^∞-function of x and for any multi-index α,

$$|\partial_x^\alpha V(x)| \leqq C_\alpha(1+|x|)^{m-|\alpha|}, \quad x\in\boldsymbol{R}^n. \tag{1.1}$$

Then the operator $\tilde{\mathfrak{H}}$ with the domain $\mathfrak{D}(\tilde{\mathfrak{H}})=\mathcal{S}(\boldsymbol{R}^n)$, the space of rapidly decreasing functions, is real symmetric in the Hilbert space $L^2(\boldsymbol{R}^n)$. We let $\mathfrak{H}$ be any one of its selfadjoint extensions and $\{E_{\mathfrak{H}}(I), I\in\mathfrak{B}^1\}$ the associated spectral measure. $\mathfrak{B}^1$ is the σ-field of Borel subsets of $\boldsymbol{R}^1$.

The purpose of this paper is to study the spectral projections $E_{\mathfrak{H}}(I)$ at high energy and to prove, in particular, the following theorem. We denote $\tilde{m}=\max(m, 2)$ and $\langle x\rangle=(1+x^2)^{1/2}$.

THEOREM 1.1. *Let $V(x)$ satisfy the assumption $(\mathrm{A})_m$, $m\geqq 0$ and let $\mathfrak{H}$ be a selfadjoint extension of $-(1/2)\Delta+V(x)|_{\mathcal{S}(\boldsymbol{R}^n)}$ in the Hilbert space $L^2(\boldsymbol{R}^n)$. Then for any $q>1/2$ and $\rho>0$ there exists a constant $C>0$ such that*

$$\|\langle x\rangle^{-q}E_{\mathfrak{H}}([\lambda-\rho\lambda^{1/2-1/\tilde{m}}, \lambda+\rho\lambda^{1/2-1/\tilde{m}}])\langle x\rangle^{-q}\| \leqq Cd_m(\lambda) \tag{1.2}$$

for all $\lambda\geqq 0$. Here $d_m(\lambda)=\langle\lambda\rangle^{-1/\tilde{m}}$ and $\langle\lambda\rangle=(1+\lambda^2)^{1/2}$.

REMARK 1.2. (1) When $V(x)\equiv 0$, it is well-known that the decay rate $\lambda^{-1/2}$ is optimal. Thus the theorem implies the invariance of the decay rate of "local spectral measure" $\langle x\rangle^{-q}E_{\mathfrak{H}}([\lambda-\rho, \lambda+\rho])\langle x\rangle^{-q}$ for $m\leqq 2$.

(2) When $V(x)$ is singular, $V\in L_{\mathrm{loc}}^p$, $p>n/2$, a weaker version of (1.2) appears in Section 6.

COROLLARY 1.3. *Let $\phi_j(x)$ be the normalized eigenfunction of $\mathfrak{H}$ associated with the eigenvalue $\lambda_j\geqq 0$. Then for $q>1/2$,*