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A pinching problem for locally homogeneous spaces

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It is well known that a Riemannian manifold is locally symmetric if and only if the curvature tensor is parallel. In 1958, W. Ambrose and I. M. Singer [1] gave a local characterization of locally homogeneous spaces, which is an extension of locally symmetric case.

In this note, we consider the following problem,

If the local property of a Riemannian manifold is similar to the above space, then is it diffeomorphic to the above?

Initiated by H. Rauch, many works for such a kind of problem are developed. Recently, M. Gromov gave a new aspect and showed a remarkable theorem. We use this theorem as a main tool. The same method is used in [2], [6], [8] etc.

For an n-dimensional Riemannian manifold (M, g), we denote by ∇ the covariant derivative, $R=R_M$ the curvature tensor, K_M the sectional curvature, $\mathrm{vol}(M)$ the volume, $\mathrm{diam}(M)$ the diameter of M. Let $\mathfrak{M}(n, \Lambda, V, D)$ be the category of all complete Riemannian manifolds M with dimension n, $|K_M| \leqq \Lambda^2$, $\mathrm{vol}(M) \geqq V$, $\mathrm{diam}(M) \leqq D$. We shall prove the following Theorems 1 and 2.

THEOREM 1. *Given integers $m, n>0$ and constants $\Lambda, V, D>0$, there is a constant $\delta=\delta(m, n, \Lambda, V, D)>0$ depending only on m, n, Λ, V, D such that if $M \in \mathfrak{M}(n, \Lambda, V, D)$ satisfies the condition that $|\nabla^m R| \leqq \delta$, then M is diffeomorphic to a locally symmetric space.*

It should be noted that Min-Oo and E. Ruh [13], [14] gave other conditions of pinching for locally symmetric space. In their assumption, the constant does not depend on the volume or the injectivity radius. In our case, the almost flat manifold is an example to show that removing the dependence of the volume is impossible. This fact is noted by Ruh.

THEOREM 2. *Given an integer $n>0$ and constants $\Lambda, V, D>0$, there is a constant $\delta=\delta(n, \Lambda, V, D)>0$ depending only on n, Λ, V, D such that if $M \in \mathfrak{M}(n, \Lambda, V, D)$ has a tensor field T of type $(1,2)$ satisfying the following conditions $(*)$, then M is diffeomorphic to a locally homogeneous space.*