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# Coisotropic calculus and Poisson groupoids

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## Introduction.

Lagrangian submanifolds play a special role in the geometry of symplectic manifolds. From the point of view of quantization theory, or simply a categorical approach to symplectic geometry [**Gu-S2**], [**W3**], lagrangian submanifolds are the "elements" of symplectic manifolds. Since the canonical transformations between symplectic manifolds $P_1$ and $P_2$ are those whose graphs are lagrangian in $P_2 \times P_{\bar{1}}$ (the "$^-$" indicating that the symplectic structure on $P_1$ has been multiplied by $-1$), one calls arbitrary lagrangian submanifolds of a product $P_2 \times P_{\bar{1}}$ *canonical relations*. It turns out that, under a transversality or clean intersection assumption, the composition of canonical relations is again canonical. Thus the canonical relations can be taken as morphisms in a symplectic "category"; the quotation marks, which are present because of the difficulties raised by the transversality condition, can be removed if we restrict attention to symplectic vector spaces and linear canonical relations.

The purpose of this paper is to extend the lagrangian calculus from symplectic to Poisson manifolds, i.e., manifolds foliated by symplectic manifolds of varying dimensions. The notion of lagrangian submanifold becomes less useful in this case (it is not even so clear how to define it when the dimension of the symplectic leaves jumps), and in fact it is the *coisotropic* submanifolds which will play the essential role. A closed submanifold $C$ of a Poisson manifold $P$ is coisotropic if, for every function $f \in C^\infty(P)$ vanishing on $C$, the hamiltonian vector field $X_f$ is tangent to $C$; equivalently, the set $I_C = \{f \in C^\infty(P) \mid f_{|C} \equiv 0\}$, an ideal with respect to multiplication, is required to be a subalgebra for the Poisson bracket. The latter definition is purely algebraic and shows that the notion of coisotropic extends as far as Poisson *algebras*.

Our main results are as follows.

1) The graph of $f: P_1 \to P_2$ is coisotropic in $P_2 \times P_{\bar{1}}$ if and only if $f$ is a Poisson map.

2) Under suitable clean intersection assumptions, the composition of coiso-

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