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Homogeneous hypersurfaces in Kähler C-spaces with $b_2=1$

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Introduction.

A compact simply connected homogeneous Kähler manifold is called a *Kähler C-space*. Recall that a Kähler C-space Y with $b_2(Y)=1$ can be obtained by a possible pair $(\mathfrak{g}, \alpha_r)$ of a complex simple Lie algebra $\mathfrak{g}$ and a simple root α_r of $\mathfrak{g}$ (cf. Section 1 below). Moreover, since $b_2(Y)=1$, the Picard group of Y is isomorphic to $\boldsymbol{Z}$. We denote its ample generator by $\mathcal{O}_Y(1)$ and $\mathcal{O}_Y(1)^{\otimes a}$ by $\mathcal{O}_Y(a)$, $a\in\boldsymbol{Z}$. For a positive integer d, a member of the linear system $|\mathcal{O}_Y(d)|$ is called a hypersurface of degree d in Y.

We sometimes encounter the phenomena that a certain Kähler C-space can be embedded in another Kähler C-space with $b_2=1$ as a hypersurface. For example, an n-dimensional projective space $\boldsymbol{P}^n$ (resp. a complex quadric Q^n) can be embedded in $\boldsymbol{P}^{n+1}$ as a hypersurface of degree 1 (resp. 2). On the other hand, Kimura [7, II] showed that the cohomology group $H^0(T_X)$ vanishes for a smooth hypersurface X in an irreducible Hermitian symmetric space of compact type if the degree of X is greater than two. This gives us the feeling that the above phenomena can be completely classified. In fact, we show the following:

MAIN THEOREM. *Let Y be a Kähler C-space with $b_2=1$. Then a Kähler C-space X can be embedded as a hypersurface of degree d in Y, if and only if X, Y and d are one of the following (up to isomorphism):*

(1) $X=\boldsymbol{P}^n$, $Y=\boldsymbol{P}^{n+1}$ *and* $d=1$.
(2) $X=Q^n$, $Y=\boldsymbol{P}^{n+1}$ *and* $d=2$.
(3) $X=Q^n$, $Y=Q^{n+1}$ *and* $d=1$.
(4) $X=(C_l, \alpha_2)$, $Y=(A_{2l-1}, \alpha_2)$: *the grassmannian* Grass(2, 2l), *and* $d=1$.
(5) $X=(F_4, \alpha_4)$, $Y=(E_6, \alpha_1)$: *the irreducible Hermitian symmetric space of type* EIII, *and* $d=1$.

In each of the above five cases, it is known that X can be embedded in Y as a hypersurface of the prescribed degree. The first three are standard and the last two examples (4) and (5) are due to Sakane [14] and Kimura [8], respectively. Thus the proof is reduced to showing the converse.