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Weak expectations in C^*-dynamical systems

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1. Introduction.

Attempts to extend a factorial state φ on a C^*-algebra B to a factorial state on a larger C^*-algebra A mainly centred around searches for solutions of a tensor product problem, or equivalently for weak expectations for the GNS representation π_φ, that is, linear contractions P of A into $\pi_\varphi(B)''$ such that $P|_B=\pi_\varphi$ (see [1] and the references cited therein). The eventual solutions of the problem [7, 9] were variants of this method.

In the case when there is an action α of an amenable group G on A leaving B invariant, an analogous problem is to consider an α-invariant state φ of B which is centrally ergodic in the sense that

$$\pi_\varphi(B)''\cap\pi_\varphi(B)'\cap u_\varphi(G)'=\boldsymbol{C}\cdot 1,$$

where (π_φ, u_φ) is the associated covariant representation of (B, G, α), and to try to find an extension to a centrally ergodic state of A. It was shown in [3] that this can be done by the method of [1] if B is (semi)nuclear, but the von Neumann algebra theory developed in [7, 9] is not sufficient to provide a general solution. A corollary of a successful solution is that if A is separable and G-central (and B is nuclear), then B is also G-central.

The purpose of this paper is to clarify the covariant situation. Firstly, in Section 2, we consider the problem lifted to the C^*-crossed products. Thus the existence of a weak expectation $\hat{Q}$ for the representation $\pi_\varphi\times u_\varphi$ of $A\times_\alpha G$ (with respect to the subalgebra $B\times_\alpha G$) is seen to be equivalent to the existence of a (covariant) completely positive contraction Q of A into $(\pi_\varphi(B)\cup u_\varphi(G))''$ such that $Q|_B=\pi_\varphi$. Under these circumstances, one may apply the results of [1] to the crossed products. Secondly, in Section 3, it is observed that, if A is G-central, then $\hat{Q}$ and Q always exist. Thus the question of G-centrality of B is reduced to the problem of arranging that Q maps A into $\pi_\varphi(B)''$.

For the theory of crossed products, the reader is referred to [8, Chapter 7]; for the basic theory of invariant states, to [4, 4.3].