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On a problem of Yamamoto concerning biquadratic Gauss sums, II

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§ 1. Introduction.

For a prime number $p\equiv 5 \pmod 8$ take positive integers a and b such that $p=a^2+4b^2$ and put $\omega=\omega_p=a+2bi$. Consider the Gauss sum

$$\tau_p = \sum_{m=1}^{p-1}\left(\frac{m}{\omega}\right)_4 e^{2\pi i m/p},$$

where $\left(\frac{m}{\omega}\right)_4$ is the biquadratic residue symbol in Gauss' number field $\boldsymbol{Q}(i)$. We write

$$\tau_p = \varepsilon_p \omega^{1/2} p^{1/4} \quad \text{with} \quad 0<\arg(\omega^{1/2})<\frac{\pi}{4}.$$

It is known that $\varepsilon_p^4=1$. Furthermore we put

$$C_p = \sum_{m=1}^{(p-1)/2}\left(\frac{m}{\omega}\right)_4.$$

For a complex number z we denote by $\bar{z}$ the complex conjugate of z and put $\mathrm{Re}(z)=(z+\bar{z})/2$ and $\mathrm{Im}(z)=(z-\bar{z})/2i$. Yamamoto [**10**] observed that the inequality

$$\mathrm{Im}(\varepsilon_p \overline{C_p}) > 0 \tag{1}$$

holds for $p<4{,}000$ and proposed the question whether this is always true. In the previous paper [7], the author reported a counter-example for (1). At the same time, it was also mentioned that there is only one counter-example for (1) up to 1,000,000. The purpose of this paper is to explain the tendency of the inequality (1) to be satisfied. We shall prove the following theorem.

THEOREM 1. *The limit*

$$\lim_{x\to\infty}\frac{\#\{p\,;\,p\leqq x,\ p\equiv 5 \pmod 8,\ \text{the inequality (1) holds for } p\}}{\#\{p\,;\,p\leqq x,\ p\equiv 5 \pmod 8\}},$$

where p denotes rational prime numbers, exists and lies between 0.9997 *and* 0.9998.

For an element μ of $\mathcal{O}:=\boldsymbol{Z}[i]$ prime to 2, denote by χ_μ the Dirichlet character modulo $2m$ induced from $\left(\frac{\cdot}{\mu}\right)_4$ where m is the smallest positive integer contained in the ideal $\mu\mathcal{O}$. Then