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Characterization of the class of upward first passage time distributions of birth and death processes and related results

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1. Introduction and main results.

We consider the minimal Markov process $\{X(t)\}_{t\geqq 0}$ on the nonnegative integers with a generator $A=(a_{ij})$ defined as follows. For nonnegative integers i and j,

$$(1.1)\qquad \begin{aligned} a_{ij} &= \beta_i && \text{if } i>0 \text{ and } j=i+1,\\ &= -(\beta_i+\delta_i) && \text{if } i>0 \text{ and } j=i,\\ &= \delta_i && \text{if } i>0 \text{ and } j=i-1,\\ &= 0 && \text{otherwise,} \end{aligned}$$

where $\delta_1\geqq 0$, $\delta_i>0$ for $i=2, 3, \cdots$, and $\beta_i>0$ for $i=1, 2, \cdots$. Such a process is called birth and death process. This process is strongly Markov by its minimality. Note that if $X(s)=0$ for some instant $s>0$, then $X(t)=0$ for all $t>s$, that is, the state 0 is a trap. Also note that the state 0 is attained from other states with positive probability whenever $\delta_1>0$. Let

$$\tau_n(\omega) = \inf\{t>0\,;\, X(t, \omega)=n\}$$

be the first passage time for $X(t)$ to n. Here we do not define $\tau_n(\omega)$ if $\{t\,;\, X(t, \omega)=n\}=\emptyset$. Let μ_{mn} be the distribution of τ_n when the process starts at m. We denote by $\sigma_{mn}(s)$ the Laplace transform of μ_{mn}, that is,

$$\sigma_{mn}(s) = E_m(e^{-s\tau_n}) = \int_0^\infty e^{-st}\mu_{mn}(dt).$$

Note that in the case $\delta_1>0$, the total mass of μ_{mn}, $1\leqq m<n$, is less than 1. We set $\bar{\mu}_{mn}=\mu_{mn}/\mu_{mn}([0, \infty))$ and $\bar{\sigma}_{mn}(s)=\sigma_{mn}(s)/\sigma_{mn}(0)$. Main purpose of this paper is to determine the class of μ_{mn}, $m<n$, for all birth and death processes.

Let $\boldsymbol{R}_+=[0, \infty)$. Let $\mathcal{P}(\boldsymbol{R}_+)$ be the totality of probability measures on $\boldsymbol{R}_+$. For $\mu\in\mathcal{P}(\boldsymbol{R}_+)$, we denote by $\mathcal{L}\mu(s)$ its Laplace transform. Let G be a pro-

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